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THE

ELEMENTS

OF

PLANE AND SOLID GEOMETRY.

BY

H. W. WATSON, M.A.

Sometime Fellow of Trinity College, Cambridge
Late Assistant Master of Harrow School.

SECOND EDITION.

LONDON:
LONGMANS, GREEN, AND CO.
1872.

PREFACE

TO

THE SECOND EDITION.

THE present edition of this work is of necessity little more than a reprint of its predecessor. Mistakes have been corrected and omissions supplied, but the nature of the subject does not admit of any very important changes either in substance or arrangement.

There have been many criticisms upon the First Edition, and these for the most part most favourable and encouraging. Among the rest, special acknowledgments are due to the writer of the notice in the *Philosophical Magazine* for October of last year, for pointing out a grave blunder in the section on the length of curved lines, whereby a proposition has been described as a definition. In drawing attention to this mistake, the writer of the notice adds some observations upon the general subject, which appear to call for a little detailed consideration here. He appears to make light of the difficulty involved in the

accurate definition of the length of a curved line, and suggests the introduction of the conception of a flexible line pulled straight. Practically we are, without doubt, familiar enough with the conception of the length of a curved line, but familiarity with a conception does not diminish the difficulty of accurate definition; on the contrary, it frequently enhances that difficulty. In the case before us, suppose the reviewer's suggestion adopted, and the length of a curved line defined as being that of the straight line with which it coincides when pulled out. In the process of pulling out the identity of form is destroyed, and nothing but identity of length remains, that is to say we must provide that in the process of rectification the *length of the curve* is to remain unaltered, and thus the thing to be defined (viz. the length of the curve) is introduced into the definition of itself. The suggested definition does, in fact, tacitly assume the conception of a curve as composed of infinitesimal rectilinear elements, and conceives that the rectification takes place by turning each element in succession round one extremity until it comes into the same straight line with the last preceding element. If such a conception of a curve be admitted, any definition of its length is, of course, superfluous, being included in the introductory definitions of the addition of straight lines, but such a conception cannot well find place in an elementary work like the present.

Another and less friendly criticism, the only one in fact which has appeared of a markedly unfavourable character, pronounces the demonstrations to be cumbersome and tedious, and especially adduces Props. 11 and 17 of Bk. I. in illustration of this assertion.

The general criticism does not, of course, admit of refutation; it must be left to the judgment of the public. But concerning Prop. 11, Bk. I., it may be well to state, what should indeed have been mentioned in the Preface to the First Edition, that the present longer demonstration was intentionally preferred to the shorter one based on the principle of continuity, and that because this principle, though most obvious and natural to advanced geometricians, is by no means so to boys and learners generally. It was thought therefore desirable to make the demonstrations in Bk. I. depend as much as possible upon the principle of superposition alone, and not to introduce the idea of continuity and continuous motion until later on in the Second Book.

BERKSWELL RECTORY:
October 1872.

PREFACE

TO

THE FIRST EDITION.

THE study of Elementary Geometry (at least in England) has been for a long time identified with one particular treatise, accepted as a standard. At the present moment there is a wide-spread dissatisfaction with that treatise; but there is very little agreement as to the manner in which it may be best improved. The most suitable Preface, therefore, to a new work on Geometry would appear to consist in an enumeration of the main features of its agreement or disagreement with Euclid, and an attempt as far as possible to justify such agreement or disagreement in each particular instance.

I. The Work agrees with Euclid in retaining the syllogistic form throughout. Many objections, strong and ably urged, have been alleged against this method of treatment by modern writers. It is said that the study of Geometry for its own sake is thereby made subordinate to its study as a logical discipline; and that the detailed syllogistic form into which all the demonstrations are thrown is a source of obscurity to beginners, and damaging to true geometrical freedom and power.

Now, whatever truth there may be in these charges as applied to Euclid's treatise, they do not appear to be applicable to the syllogistic form of statement in itself. No doubt there is very much unnecessary prolixity in some of Euclid's demonstrations. He has in many cases perversely refused to draw the most general inference possible from his premises, even where by doing so he would not have lengthened or embarrassed his reasoning, and the result has been much useless repetition, and vain expenditure of words. As one illustration out of very many which might be adduced of the truth of this assertion the 26th Proposition of Book I. may be instanced. The demonstration of this proposition would have been in no degree lengthened if Euclid had extended it to the equality of the triangles in all respects. As it is, he has restricted himself to the sides and angles, and accordingly, when in Proposition 34 he has proved the equality of the opposite sides and angles of a parallelogram by means of this Proposition 26, he is obliged to recur to Proposition 4, with its long and cumbrous enunciation, to arrive at the equality of the areas of the two triangles into which the diameter divides the parallelogram.

Nothing can be said for a waste of time and words like this, but it would be unfair to charge the syllogistic method generally with such an obvious defect in its application. The truth seems to be that the syllogistic method, if properly applied, may be made

as concise and intelligible as any other, and it has one important claim to be retained in an elementary geômetrical treatise, inasmuch as there is absolutely no place for strict logical treatment in any other part of our modern mathematical course. Analysis, even in its simplest form soon, and of necessity, degenerates into a manipulation of symbols. The elements of Algebra must be acquired as a mechanical exercise, as an art rather than a science. The very rudiments of algebraical analysis involve difficulties far beyond a schoolboy's intellect, and necessitate as much profound thought as the highest mathematics. In Geometry, on the other hand, the conceptions involved are plain and easy, 'every step is planted on firm ground,' and the vigorous logical treatment of which these conceptions are susceptible not only supplies a mental discipline of inestimable educational value, but furnishes a test whereby to try the conclusions arrived at by the more rapid processes of analytical reasoning.

II. While, however, the syllogistic form has been retained in this treatise for the reasons which have been mentioned, innovations have been introduced in the following important respects:

1. An extended application of the principle of superposition.

The application of this principle is the great stumbling-block to the young geometrician, and yet it is absolutely indispensable, and hardly a step

can be made without it. Euclid has recognised its importance, and introduced it early enough in his course; but he uses it grudgingly and timidly, and he imparts this feeling of timidity to the learner. Thus, in Proposition 4 of Book I. there is no suggestion of the possibility of the two triangles being so situated as to make it necessary for the plane of one of them to be reversed before superposition can take place; the proof adopted obviously supposes a simple transference of position, the same face remaining uppermost throughout. It is true that the language employed is sufficiently general to include every case, but it is equally true that the ideas suggested to the reader are limited in the manner that has been mentioned. Now in Proposition 5, which is the very first instance of the application of Proposition 4, the triangles are so situated as to require this reversal of the plane of one of them before they can be superposed, and the whole of the long and tedious reasoning of this proposition might have been avoided if that had been done at first explicitly and avowedly which is afterwards done substantially, though in a sense surreptitiously. To avoid the superposition of a triangle upon itself with plane reversed, Euclid has recourse to the artificial and almost disingenuous device of turning one triangle into two by producing the sides, and then applying Proposition 4 to these two triangles, ignoring the fact that the results of Proposition 4 are really

being applied to a case for which the proposition has never been explicitly proved. Since the application of the test of superposition is so very indispensable, it would seem desirable to give as much prominence to it as possible early in the course; thus, by frequent use, rendering it familiar to the learner, and thereby divesting it of its terrors. This method has accordingly been adopted in the following treatise.

2. The introduction of hypothetical constructions.

Euclid's Geometry comprises two classes of propositions, entirely different in their aim and nature, but mixed up together without any indication of this difference. The propositions in the one class are theoretical and general, while those in the other are practical and special. The former treat of the *science of Geometry*, the latter of the application of that science to the *art of geometrical drawing*. There is no doubt that great light may be thrown upon, and additional interest imparted to, the theorems by the problems; but this advantage is counterbalanced by the disadvantage attending Euclid's treatment. The strictly logical form adopted by him induces, and was intended to induce, the belief that each individual proposition is essential to all that follow. Hence, an inevitable confusion arises in the mind of the reader between that which is possible theoretically and conceivably, and that which is possible in relation to the instruments to which Euclid chooses to restrict him-

self, i. e. the compasses and ruler. Many advanced mathematicians even would be puzzled to give an explanation offhand of the impossibility of the time-honoured problems of squaring a circle, bisecting a cube, and trisecting an angle. Nor is this confusion of ideas the only evil result arising from the exclusion of hypothetical constructions, for the treatise is thereby rendered inconsistent with itself, and properties of the circle are assumed in Book I. which are demonstrated at full length in Book III. (compare, for example, Book I. Proposition 12 with Book III. Proposition 2); and in Book XI. recourse is necessarily had to hypothetical constructions, where lines are supposed to be drawn in space, concerning which all that is known is the possibility of the existence of such lines.

3. The arithmetical treatment of ratio and proportion.

The unanswerable objection to Euclid's treatment of ratio and proportion is that it is practically disregarded. 'The reasoning is exquisite and profound, it is too exquisite,' it is artificial and remote from our practical common-sense notions on the subject. No teacher dreams of taking his pupils through Euclid's fifth book; and thus the opportunity for acquiring much valuable instruction is for ever lost to the student. It is at this point that the conception of number is properly brought into contact with the conception of continuous magnitude, and no arithmetical treatise on fractions

can adequately supply the omission. In the following treatise the properties of ratio and proportion are, in the first place, explained and proved in a few simple propositions with reference to commensurable magnitudes, and they are afterwards extended by the simple application of the method of limits to incommensurable magnitudes.

4. The admission of axioms even where derivable from other axioms already stated (as Axiom 2, upon straight lines).

It does not seem to be of any great importance what truths are assumed as axiomatic (i.e. whether demonstrable or not), provided they really have that character and require no demonstration to make them clear to the mind of the learner; but it is important that they should not be too numerous, and also that they should be distinctly enunciated as axioms and not tacitly assumed from time to time as intuitions. Many Continental and some English writers appear to be somewhat lax in this respect. They attach so much importance to the rapid acquisition of mere geometrical knowledge that they lose sight of the equal if not greater importance of obtaining a mastery of the processes by which such knowledge may be attained. Without doubt the clearness of intuition acquired by a practised geometrician will frequently make him impatient of the successive steps of detailed reasoning, and he will be eager to conduct his pupil to the desired end by a shorter and an easier

road. The learner, too, yields a ready acquiescence in the conclusion arrived at; but it is a great misfortune for him if he is deprived of that useful discipline in geometrical reasoning which, however tedious and clumsy it may seem to be in its application to short and easy propositions, is indispensable to the investigation of those which are more intricate and difficult. As a matter of fact two demonstrable truths, and two only (Axioms 2 and 3), have been enunciated as axioms in this Treatise, and of these Axiom 3 (of the equality of all right angles) has been also assumed by Euclid, though admitting of proof by the principle of superposition.

5. The introduction of such terms and conceptions as *a limit; a moving point, line, and plane; locus; projection; intersection of loci;* and so forth.

These terms and conceptions are of such frequent occurrence in modern mathematics that their introduction needs no apology; they are perfectly definite and simple, and there is, in fact, hardly any elementary treatise, either in England or on the Continent, in which they do not appear.

III. It remains to notice a few less essential matters bearing chiefly on form and arrangement.

1. All the propositions concerning areas have been grouped together in a book by themselves (Book IV.).

Most teachers must have perceived an inexplicable difficulty in the minds of learners when dealing with

the intrinsically easy propositions of Euclid's Book II. and those propositions of Book I. in which areas are mentioned. By the method adopted in the following treatise it is hoped that these propositions may mutually illustrate and explain each other, and that their scope and object may be more clearly perceived by the student.

2. Many abbreviations and symbols have been adopted, but great care has been taken to secure their being regarded as abbreviations for words and phrases, and not as representing operations.

It is impossible to be too particular in this matter, even at the risk of appearing pedantic. The processes of Algebra depend upon the *laws* of operation and those laws only, the operations themselves being as frequently as not uninterpreted. In Geometry, on the other hand, every statement has a definite and ascertained meaning. The operations of Algebra soon become so easy and natural that when symbols are used it is almost impossible to avoid working by mechanical processes only in cases where geometrical reasoning, dealing with defined operations, would require distinct proof. (See the Note to Proposition 6, Book V. p. 153.)

From the same anxiety to keep a definite object of thought, rather than a mere symbol, present to the mind of the student, care has been taken to avoid as much as possible the use of symbols to represent abstract magnitudes. Therefore in the propositions

of ratio and proportion, where magnitudes are dealt with in the abstract, the demonstrations are given for certain magnitudes as lines, areas, and so forth; and the same demonstrations are then shown to be applicable to all kinds of magnitudes whatever.

3. As it is desirable that an elementary treatise should be as independent as possible of external aid, an attempt has been made to assist the self-taught student by means of notes; these notes are inserted from time to time in the course of the work, because they are intended to be read; and not grouped together at the end, in which case they would certainly be neglected. They generally contain an *informal* introduction of some new conception which is stated *formally* in a definition either preceding or following.

4. Axiom 2 concerning straight lines is usually stated thus:

A straight line is the shortest distance between any two points in it.

The statement in the text has been adopted because the length of any other than a straight line requires to be *defined* before it can be referred to in a definition or axiom.

The definition of the length of a broken line is easy and simple, and it is given at the commencement of the work, but the definition of the length of a curved line has been postponed to the end of Book II., because it involves conceptions likely to be perplexing to a beginner.

5. This treatise lays claim to very little originality. Many of the demonstrations have been adapted from existing works on geometry, especially the treatise by Messrs. Rouché and Comberousse.

Also many very valuable suggestions and amendments are due to friends of the author, and especially to D. D. Heath, Esq., formerly Fellow of Trinity College, Cambridge; and to Professor Goodeve, and R. B. Haywood, Esq., formerly Fellows of St. John's College, Cambridge.

BERKSWELL RECTORY:
June 1871.

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GEOMETRY

INTRODUCTION.

SECTION I.—PRELIMINARY DEFINITIONS AND AXIOMS.

ALL bodies, that is to say, all material objects, exist in space and possess length, breadth, thickness, form, and position.

The science of Geometry deals with the properties of bodies in connection with the space occupied by them, that is to say, their form, magnitude, or position, and does not deal with any other qualities they may possess, such as colour, texture, hardness, weight, and so forth.

Def. 1.—A *solid*, or *solid space*, is the portion of space which is or may be occupied by any body. A solid has *length*, *breadth*, *thickness*, *form*, and *position*.

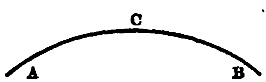
Def. 2.—A *surface* is the boundary of a solid. A surface has *length*, *breadth*, *form*, and *position*, but no *thickness*.

Def. 3.—A *line* is the boundary of a surface. A line has *length*, *form*, and *position*, but neither *breadth* nor *thickness*.

Def. 4.—A *point* is the boundary or extremity of a line. A point has *position*, but neither *length*, *breadth*, *thickness*, nor *form*.

A point is usually denoted by a single letter, and a line by two or more letters, of which the first and last denote the extremities of the line.

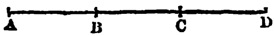
Thus, A, C, and B in the annexed figure denote points, and a line having the points A and B for its extremities and the point C situated upon it, is denoted by AB or ACB.



Def. 5.—A *straight* line is a line of such a form that if any one portion of it be applied to any other portion so that the extremities of the one portion coincide with those of the other, then every intermediate point of the one portion *must* coincide with some intermediate point of the other.

Def. 6.—Two straight lines are said to be *equal*, or of *equal* length, when the extremities of the one line can be made to coincide respectively with the extremities of the other.

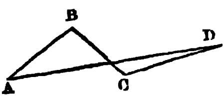
Def. 7.—Straight lines are *added* together by placing them one after another in succession in the same straight line so that one extremity of each newly added line coincides with one extremity of the last added line.*



Thus AB, BC, and CD are added together and form the straight line AD.

AB, BC, and CD are called the *parts* of AD, and AD is called the *sum* of AB, BC, and CD.

Def. 8.—A *broken* line is formed of two or more straight lines united at their extremities but *not* in the same straight line.



Thus AB, BC, and CD form the broken line ABCD.

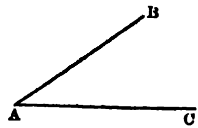
The extremities of any straight line or any broken line are said to be *joined* by the straight line or broken line, and the length of the broken line is the length of the sum of its component straight lines.

All other lines besides straight lines and broken lines are either *curved* lines or *mixed* lines.

Def. 9.—A *plane* surface is one in which any two points being taken the straight line which joins them lies wholly in that surface.

* For perfect accuracy the definition should conclude with some such sentence as the following, 'provided that no portion of any newly added line coincides with any portion of the last added line.'

Def. 10.—If two straight lines in one plane have a common extremity, and either one of them be turned round this common extremity until it coincides with the other, the revolving line being always in the same plane, the ***smallest amount*** of ***turning*** required to effect this coincidence is called the ***angle between the two straight lines.***

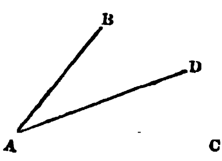


Thus the smallest amount of turning about A required to bring either of the straight lines AB or AC into coincidence with the other, the revolving line being always in the same plane, is called the angle between the lines AB and AC.

The angle in the last figure is sometimes denoted by the single letter A and sometimes, for greater distinctness, by the three letters BAC or CAB (A being always in the middle). The point A is called the *vertex*, and the straight lines AB and AC are called the *arms* of the angle, and these lines are said to *contain*, or *include*, or *form* the angle BAC.

Def. 11.—Angles are said to be equal when they can be placed one upon the other in such a way that the vertex and arms of the one can be made to coincide with the vertex and arms of the other.

Def. 12.—Angles are said to be *adjacent* when they have a common vertex and one common arm, their non-coincident arms lying on opposite sides of the common arm.

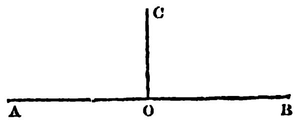


Thus the angles BAD and CAD are adjacent angles, of which AD is the common arm.

Def. 13.—Angles are *added together* by placing them so as to be adjacent to each other: thus the angle BAC in the figure is called the *sum* of the two angles BAD and CAD.

Def. 14.—When two adjacent angles are *equal* to each other the common arm is said to *bisect* the angle between the two non-coincident arms.

Def. 15.—When two adjacent angles are equal to each other and the two non-coincident arms are in the same straight line, each of the equal adjacent angles is called a *right* angle, and the common arm is said to be *perpendicular* or *at right angles* to the straight line in which the non-coincident arms lie.



Thus if AOB be a straight line and OC a line such that AOC is equal to BOC, then each of the angles AOC and BOC is called a right angle, and OC is said to be perpendicular or at right angles to AB.

Def. 16.—A *triangle* is a closed figure contained by three finite straight lines, which are called its *sides.*

Def. 17.—A triangle is called *isosceles* when two of its sides are equal, *equilateral* when all three sides are equal, and *equiangular* when all three angles are equal.

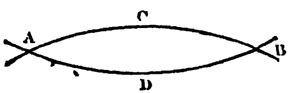
Def. 18.—When one of the angles of a triangle is a right angle, the triangle is said to be *right angled,* and in this case the side opposite to the right angle is called the *hypothenuse.*

Def. 19.—A triangle is sometimes regarded as standing upon one of its sides, which is then called its *base*; and the angle opposite to that side is then called the *vertex.*

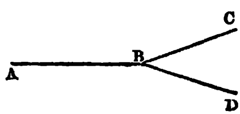
When a truth is so obvious as to be manifest without demonstration it is called an axiom. The following statements are axioms.

Axiom 1.*—Two straight lines cannot intersect in more than one point.

1.



2.



* This axiom may be otherwise stated as follows :—
If two straight lines have more than one point in common they lie in one and the same straight line.

Thus in figure 1, the lines ACB and ADB cannot both of them be straight lines.

And in figure 2, the lines ABC and ABD cannot both of them be straight lines.

Axiom 2.—The length of the straight line joining any two points is less than the length of any broken line whatever joining the same two points.

Axiom 3.—All right angles are equal to one another.

A *theorem* is a geometrical truth capable of demonstration by reasoning from certain known truths.

A *corollary* to a theorem is a geometrical truth easily deducible from that theorem.

The sign = represents equality and is an abbreviation for the words *is equal to*, or *are equal to*, according as one or more things are referred to.

The sign + expresses addition and ~ subtraction.

Thus A + B is to be read as *A together with B*, or *the sum of A and B*, and A ~ B is to be read as *the difference of A and B*, or the *remainder after subtracting the less of the two magnitudes A and B from the greater.* The phrase *A and B together* is sometimes used for *the sum of A and B.*

We sometimes establish the proof of a theorem by assuming that it is not true, and proving that such assumption leads to an inconsistency or an impossibility, as for instance that one magnitude is both greater than and equal to another magnitude, and so on.

A proof of this kind is called an *indirect* proof, or a *reductio ad absurdum.* See for example Bk. I. Prop. 4.

The assumption, whether true or false, upon which any argument is based is called the *hypothesis.*

SECTION II.—NOTES TO THE PRECEDING DEFINITIONS AND AXIOMS.

1. The conceptions of a surface line and point may be assisted by the following illustrations :

Suppose a portion of space to be completely filled by two bodies, as a block of wood and a block of stone, of any shapes respectively, provided they exactly fit into each other, so that within the space in question there is no portion which is not completely filled up, then it is clear that there is a region within this space which belongs as much to the wood as to the stone.

Such a region is called a *surface* or *superficial space.*

This surface can have no thickness, for if it had a thickness ever so small points might be found in it belonging entirely to the wood or to the stone, and such points could not, therefore, be situated on the surface.

Again, the block of wood might be situated alone in space, and then there would be a region belonging as much to the wood as to the surrounding space, and this region would be the bounding surface of the block of wood, or, to speak more correctly, of the portion of solid space occupied by the block of wood.

Again, we might suppose two surfaces in juxtaposition, the one coloured red and the other green, and it is clear that there would, in this case, be a region belonging as much to the red as to the green surface: such a region is called a line.

A line can have no thickness because it is a portion of a surface, and it may be made evident that a line can have no breadth, by reasoning similar to that employed to show that a surface can have no thickness.

If two lines intersect, any position in space common to both of them is called a point.

A point can have neither breadth nor thickness, inasmuch as it is situated on a line, and it may be shown to have no length, by reasoning similar to that employed to show that a surface has no thickness or a line no breadth.

2. All material objects must have length, breadth, and thickness; and therefore no body can be found which occupies such a region of space as a surface or a line; never-

theless we can *conceive* such objects as existing, and we call them *material surfaces*, or *lines*.

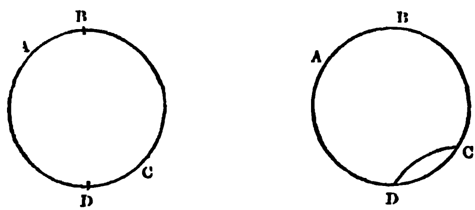
A sheet of paper has length and breadth, each so much greater than its thickness that we come almost unconsciously to regard it as having length and breadth only, that is, as a material surface; if, however, it were viewed through a microscope the thickness would become apparent.

Gold may be hammered out into leaves of such exquisite tenuity that many thousands, laid one upon the other, would not, in the aggregate, have the thickness of an inch: each of these leaves suggests the idea of a material surface more forcibly than a sheet of paper does.

So a piece of fine string suggests the idea of a material line, a piece of thread does this yet more forcibly, and a spider's web more forcibly than either.

3. We may form the idea of a *material straight line* from a piece of thread or fine wire tightly stretched, and the portion of space occupied by such a body suggests the idea of a straight line.

It is important to observe the force of the word *must* in the definition of a straight line, for there is a certain curved line called a circle, of which it would be equally true that if one portion were applied to any other portion, so that the extremities of the one portion coincided with the extremities of the other portion, then every intermediate point of the one portion *might* coincide with an intermediate point of the other.



Thus, the portion AB of the circle ABC may be applied

to the portion CD so that the extremities and every intermediate point of AB coincide with the extremities and every intermediate point of CD.

But AB *may* also be applied to CD in such a way that A and B coinciding with C and D, the intermediate points of AB, do not coincide with the intermediate points of CD, as in the second of the two figures drawn above.

4. An idea of an angle may be gained by observing the hands of a watch, each hand being supposed to be indefinitely thin, or to be a material line. If these hands occupied certain given positions, and one of them were turned by means of the key until it came into the position immediately overlying the other, the smallest amount of turning required to effect this coincidence would be the angle between the two hands.

Suppose the long hand pointed to XII and the short hand to II, then we see that the amount of turning required to bring one hand into the position overlying the other is the same as it would be if the long hand were at XII and the short hand at X, and we should say that the angle between the hands in the first case was equal to the angle in the second case.

Again, if the long hand were at XII and the short hand at IV, we see that the amount of turning requisite in this case would be twice as great as that required in either of the last two cases, and therefore we should say that the angle between the hands in this case was twice as great as that in either of the former cases.

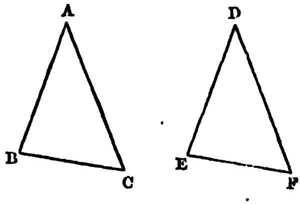
It is evident that the length of the hands has nothing to do with the magnitude of the angle between them.

5. The size and shape of a body may remain the same, when the body is transferred from one position to another in space.

By this means we frequently compare one portion of space with another.

For example, suppose that a *material surface* in the shape

of a triangle first occupied the space ABC, and was then moved and made to occupy the space DEF, its shape remaining unaltered, it would follow that the portion of superficial space ABC was, as to shape and size, exactly identical with the portion DEF.



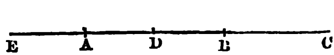
In such a case as this we often speak of transferring the triangle ABC to the triangle DEF; such language is really incorrect, inasmuch as a triangle, a line, &c., indicate fixed portions of space.

What is really transferred is the *material surface* which at one time exactly occupied the portion of superficial space indicated by the triangle ABC, and was afterwards made to occupy the space DEF.

This method has been employed to compare the magnitudes of lines and angles in Defs. 6 and 11 supra.

6. When we know that a certain line, or point, or surface must exist, we shall assume that we have found this line, or point, or surface, and reason upon such assumption accordingly.

Thus if A and B be any two points, we shall assume that a line may be drawn, having A and B for its extremities, and drawing such a line is called *joining AB.*



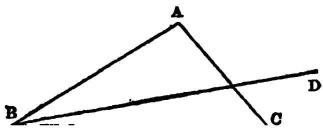
So, again, if AB be a *finite* straight line, we shall assume that straight lines, as BC or AE, may be drawn of any length, and making one straight line with AB.

The drawing such lines is called 'producing AB towards B or towards A respectively.'

Again, we assume that a point D may be found which is the middle point of the straight line AB, and the finding

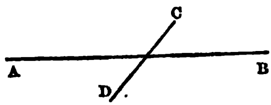
such a point we call 'bisecting AB in D.' And so on in many similar cases.

7. Besides the axioms already explicitly referred to, there are many other geometrical truths assumed by us as too obvious to require express mention. E. g.:

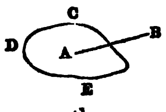


If the indefinite straight line BD were drawn between the sides AB and BC of the triangle ABC, as in the annexed figure, we should take it for granted that the straight line BD *must* meet the side AC in some point.

And if C and D were two points situated on opposite sides of the straight line AB of indefinite length, we should take it for granted that the straight line CD must meet the line AB.



Again, we assume it to be self-evident that if a point A within a closed figure DCE be joined to a point B without that figure, then the joining line must meet the boundary of the figure in some point; and similarly in many other cases.



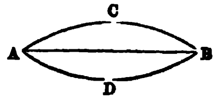
If we anticipate the definitions of the length of a curved line and of the equality of curved lines which are given hereafter in the course of Book II., and which have been postponed with the view of simplifying the commencement of the subject as much as possible, we may enunciate the second axiom of the text in the following more general language, namely:

The straight line joining any two points is the shortest distance between these points.

We may also prove that this is not really an independent statement, but that it follows from the first axiom and the assumption that one and only one line always exists which is shorter than any other line between two given points.

The proof will stand thus:

Let A and B be any two points, and AB the straight line which joins them, then AB shall be the shortest distance between the points A and B.



Since there must be some line which is the shortest distance between A and B, if this line be not AB let it, if possible, be ACB.

Let the line ACB be turned about AB until it coincides with some other line as ADB.

Because ACB may be made to coincide with ADB, the length of ACB must be equal to the length of ADB.

Because ACB is the shortest distance between the points A and B the length of ACB is less than that of ADB.

Therefore the length of ACB is both equal to and less than that of ADB, which is impossible.

Therefore ACB is not the shortest line between A and B, and in the same way it may be proved that no line except AB can be the shortest line between A and B. Therefore AB is the shortest line between A and B.

BOOK I.

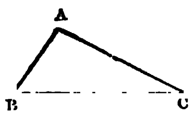
SECTION I.—ON TRIANGLES.

PROPOSITION 1.

Any one side of a triangle is less than the sum and greater than the difference of the two remaining sides.

Let ABC be a triangle, then any one of its sides as BC shall be less than the sum, and greater than the difference, of the two remaining sides AB and AC.

Fig. 1.



Because the two points B and C are joined by the straight line BC and the broken line BAC,

therefore the length of BC is less than that of BAC (*Ax.* 2),

that is, BC is less than BA+AC.

Again, suppose that AB is not greater than AC,

Then AB+BC is greater than AC by what has been already proved,

take away AB from each,

therefore BC is greater than AC ~ AB.

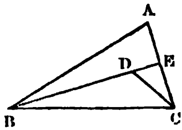
PROPOSITION 2.

If from the ends of a side of a triangle there be drawn two straight lines to a point within the triangle, these two lines shall be together less than the two remaining sides of the triangle.

From B and C the ends of the side BC of the triangle ABC, let the two straight lines BD and CD be drawn to

the point D within the triangle, then the two lines BD and CD shall be together less than the two sides BA and AC.

Fig. 2.



Produce BD to meet the side AC in the point E.

Because any side of a triangle is less than the sum of the two remaining sides,

therefore BE is less than BA+AE;

add EC to each,

therefore BE+EC is less than BA+AC.

By similar reasoning it may be proved that

BD+DC is less than BE+EC;

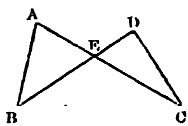
therefore BD+DC is less than BA+AC.

PROPOSITION 3.

If upon the same base and the same side of it there be two triangles having the vertex of each without the other, the sum of the two sides which intersect shall be greater than the sum of the two sides which do not intersect.

Upon the same base BC, and upon the same side of it, let there be two triangles ABC and DBC, having each of their vertices A and D outside of the other triangle, then the sum of the two intersecting sides AC and DB shall be greater than the sum of the two non-intersecting sides AB and DC.

Fig. 3.



Let DB and AC intersect in the point E.

Because two sides of a triangle are together greater than the third side,

therefore AE+EB is greater than AB.

Similarly CE+ED is greater than CD,

therefore AE+EB+CE+ED is greater than AB+CD.

But AE+CE is equal to AC,

and EB+ED is equal to DB,

therefore AC+DB is greater than AB+DC.

PROPOSITION 4.

Upon the same base and on the same side of it there cannot be two triangles having the two sides terminated in one extremity of the base equal to each other, and at the same time the two sides terminated in the other extremity equal to each other.

Fig. 4.

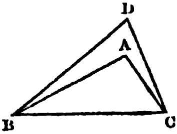
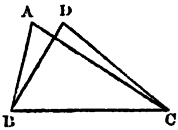


Fig. 5.



If possible, upon the same base BC, and on the same side of it, let there be two triangles ABC and DBC, having the two sides BA and BD, terminated in B, equal to each other, and at the same time the two sides CA and CD, terminated in C, equal to each other.

First, let the vertex A of one of the triangles fall within the other triangle, as in Fig. 4.

Because BA and CA are drawn from B and C to the point A within the triangle DBC,

therefore BA + CA is *less* than BD + CD (Prop. 2).

But BA and CA are equal to BD and CD respectively,

therefore BA + CA is *equal* to BD + CD,

which is impossible.

Next, let the vertex of each triangle be without the other, as in Fig. 5.

Because AB is equal to DB, and DC is equal to AC,

therefore AB + DC is *equal* to DB + AC.

But DB + AC is *greater* than AB + DC (Prop. 3),

which is impossible.

Def. 20.—A triangle is said to have six *parts*—three sides and three angles, and two triangles are said to be *equal in all their parts* when the three sides and the angles opposite to them of the one are equal, respectively, to the three sides and the angles opposite to them of the other, equal angles being opposite to equal sides.

PROPOSITION 5.

If two triangles have two sides of the one equal to two sides of the other, each to each, and the angle contained by the two sides of the one equal to the angle contained by the two sides of the other, the two triangles shall be equal to each other in all their parts.

Fig. 6.

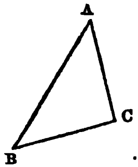


Fig. 7.

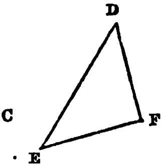
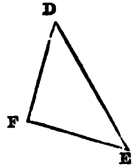


Fig. 8.



Let ABC and DEF be two triangles, having the two sides BA and AC equal to the two sides ED and DF each to each, and the angle BAC equal to the angle EDF, then shall BC be equal to EF, and the angles ABC and ACB to DEF and DFE, each to each, in the order mentioned.

Let the triangle ABC be applied to, or placed upon, the triangle DEF, so that the point A may fall upon the point D, and the straight line AB upon the straight line DE.

Then the point B shall fall upon the point E because AB is equal to DE.

Two cases now occur for consideration.

First, the triangle DEF may be situated as in Fig. 7.

In this case when AB coincides with DE, as above mentioned, AC and DF will lie on the same side of DE.

Then because AB coincides with DE, and the angle BAC is equal to the angle EDF, the side AC must coincide with the side DF.

Because A coincides with D, and AC coincides with and is equal to DF, the point C must coincide with the point F.

But the point B coincides with the point E, therefore the straight line BC coincides with and is equal to the straight line EF. (*Ax.* 1 and *Def.* 6.)

And because AB coincides with DE and BC with EF the angle ABC is equal to the angle DEF. (*Def.* 11.)

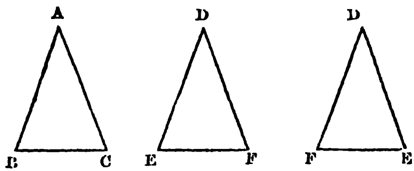
Similarly the angle ACB is equal to the angle DFE; therefore the triangles ABC and DEF are equal in all their parts.

Next let the triangle DEF be situated as in Fig. 8.

In this case when AB coincides with DE, AC and DF must lie on *opposite* sides of DE unless the plane of the triangle ABC be *reversed* so that the face of the plane below the paper be made to become the face *above* the paper.*

When this has been done, AC and DF will lie on the same side of DE, and then it may be proved, exactly as in the first case, that the triangle ABC is equal to the triangle DEF in all its parts.

Fig. 9. Fig. 10. Fig. 11.



Corollary 1.—If AB be equal to AC and therefore DE to DF, as in Figs. 9, 10 and 11, then it may be proved, as in the proposition, that the triangle ABC may, without reversing its plane in either case, be applied to the triangle DEF, in the position of Fig. 10, so that the points A, B, and C coincide with D, E, and F respectively, or to the triangle

* The statement in the text may be illustrated as follows:

Suppose the page upon which the Figs. 6, 7, and 8 are drawn to be coloured red, and the back of the page to be coloured green.

Let the triangle ABC be cut out of the sheet of paper.

Then the triangle ABC so cut out may be applied to the triangle DEF, Fig. 7, in such a way as to fit entirely into it, the red face being uppermost. But the triangle ABC so cut out could not be turned about in any way so as to fit into the triangle DEF, Fig. 8, so long as the red face of the piece of paper cut out was kept uppermost.

If however the piece of paper cut out of the triangle ABC were reversed so that the green face was kept uppermost, then the piece of paper cut out might, by turning it about in a proper manner, be made to fit exactly into the space DEF (Fig. 8).

DEF, in the position of Fig. 11, so that the points A, B, and C, coincide with D, F, and E respectively, therefore either the angle ABC or the angle ACB may be made to coincide with the angle DEF, and therefore the angle ABC must be equal to the angle ACB, therefore *the angles at the base of an isosceles triangle are equal to one another.*

Corollary 2.—If two triangles can be transferred so that their angles coincide each with each they are equal to one another in all their parts, and if they be equal to one another in all their parts they may be so transferred.

PROPOSITION 6.

If two triangles have two angles of the one equal respectively to two angles of the other, and the side lying between the two angles of the one equal to the side lying between the two angles of the other. the two triangles shall be equal in all their parts.

Fig. 12.

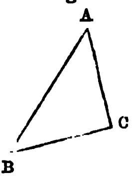


Fig. 13.

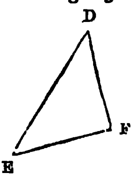
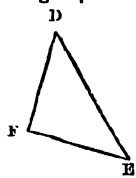


Fig. 14.



Let ABC and DEF be two triangles, having the two angles ABC and ACB of the one, respectively, equal to the two angles DEF and DFE of the other, and the side BC equal to the side EF, then shall the two triangles be equal in all their parts.

Let the triangle ABC be applied to the triangle DEF, so that the point B coincides with the point E, and the straight line BC with the straight line EF, then the point C shall coincide with the point F, because BC is equal to EF. Two cases now arise for consideration.

First, the triangle DEF may be situated as in Fig. 13.

In this case when the points B and C coincide with E and F respectively, as above mentioned, the triangles ABC

and DEF will lie on the same side of EF. Then, because BC coincides with EF, and the angle ABC is equal to the angle DEF, the side BA must coincide with the side ED.

Similarly the side CA must coincide with the side FD.

Then, because the sides BA and CA coincide with the sides ED and FD, and because two straight lines can only intersect in one point, therefore the point A must coincide with the point D.

Because A coincides with D, and B with E, therefore the side AB is equal to the side DE, and similarly the side AC is equal to the side DF.

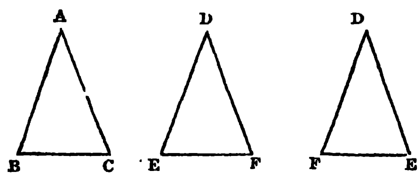
Because the straight lines AB and AC coincide with DE and DF respectively, therefore the angle BAC must be equal to the angle EDF. (Def. 11.)

Therefore the triangles ABC and DEF are equal in all their parts.

Next let the triangle DEF be situated as in Fig. 14.

In this case when the points B and C have been made to coincide with the points E and F respectively, the triangles ABC and DEF will lie on opposite sides of EF, unless the plane of the triangle ABC be *reversed* so that the face of the plane *below* the paper be made to become the face *above* the paper. When this has been done the triangles ABC and DEF will be situated on the same side of EF, and then it may be proved exactly, as in the first case, that the triangles ABC and DEF are equal in all their parts.

Fig. 15. Fig. 16. Fig. 17.



Corollary.—If the angle ABC be equal to the angle ACB, and therefore also DEF to DFE, then it may be proved

as in the proposition, that the triangle ABC (Figs. 15, 16, 17) may without reversing its plane in either case, be applied to the triangle DEF in such a way that the sides AB and AC coincide respectively with either DE and DF, in the position of Fig. 16, or with DF and DE, in the position of Fig. 17.

Therefore AB and AC are each of them equal to DE, and therefore AB is equal to AC.

Therefore *if two angles of a triangle be equal to each other the sides which subtend these equal angles shall be also equal.*

PROPOSITION 7.

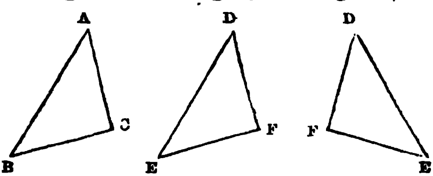
If two triangles have the three sides of one of them equal, respectively, to the three sides of the other, the triangles shall be equal in all their parts.

Let ABC and DEF be two triangles having the three sides AB, BC, and CA, of the one equal, respectively, to the three sides DE, EF, and FD of the other; then shall the two triangles ABC and DEF be equal in all their parts.

Let the triangle ABC be applied to the triangle DEF, so that the point B may coincide with the point E, and the straight line BC with the straight line EF, then the point C must coincide with the point F, because BC is equal to EF.

If the triangle DEF be situated as in Fig. 19, then when the points B and C coincide with the points E and F

Fig. 18. Fig. 19. Fig. 20.



respectively, the triangles ABC and DEF must lie on the same side of the line EF. Therefore the two sides BA and AC must coincide with ED and DF respectively, and the point A must coincide with the point D.

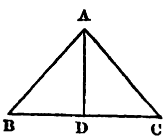
For if they did not, then upon the same base EF and upon the same side of it there would be two triangles, AEF and DEF, having the two sides AE and DE terminated in the extremity E of the base equal to each other, and at the same time the two sides AF and DF terminated in the other extremity F equal, and this is impossible by Proposition 4; therefore the side AB coincides with the side DE, and the side AC with the side DF, and therefore the angles at A, B, and C are equal to the angles at D, E, and F respectively, and the triangle ABC is in all its parts equal to the triangle DEF.

If the triangle DEF be situated as in Fig. 20, then it may be proved that, by reversing the plane of the triangle ABC as in the preceding propositions, the points A, B, and C may be made to coincide with the points D, E, and F respectively, and therefore that the triangle ABC is in all its parts equal to the triangle DEF.

Note.—There is no more instructive exercise for the student than the endeavour to prove the same proposition in various ways. By this means great insight is gained into the relation of one geometrical truth to another, and the habit is formed of contemplating the same proposition in many different aspects.

For example, we have proved in the corollaries to Propositions 5 and 6, respectively, the proposition that *the angles at the base of an isosceles triangle are equal to one another, and its converse.* We might also proceed as follows:

Fig. 21.



Let ABC be an isosceles triangle having the sides AB and AC equal, then shall the angles ABC and ACB be also equal.

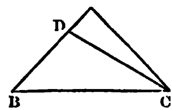
Let D be the middle point of BC and join AD.

Then the three sides of the triangle ABD are respectively equal to the three sides of the triangle ACD, and therefore the angles in the two triangles are equal, each to each.

Therefore BAD is equal to CAD, ADB to ADC, and ABC to ACB,

and the last mentioned two angles are those subtending the equal sides AB and AC.

Fig. 22.



Again, let ABC be a triangle in which the angles ABC and ACB are equal to each other, then shall the sides AB and AC be also equal.

If not, let one of them, as AB, be greater than AC, then there is some part BD of AB equal to AC ; join DC.

In the triangles ABC and DBC, the sides DB and BC are equal to AC and CB in the order mentioned, and the included angles are equal, therefore the angles opposite to the equal sides in each are equal, and therefore DCB is equal to ABC.

But ABC is equal to ACB (by hypothesis),

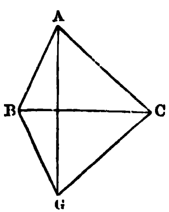
therefore DCB is equal to ACB, which is impossible,

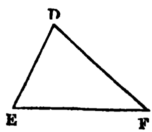
therefore the sides are equal.

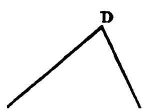
Again, we might prove Proposition 7 without the aid of Propositions 1, 2, 3 and 4 as follows :

Prove Proposition 5 by superposition, as in the text, and its corollary for the case of an isosceles triangle.

Fig. 23 Fig. 24. Fig. 25.







Let ABC and DEF be two triangles, having the three sides of one equal, each to each, to the three sides of the other.

Apply DEF to ABC, so that the points E and F coincide with the points B and C, and that the triangles DEF and ABC lie on the opposite sides of BC, reversing the plane for Fig. 24.

Let G be the position of D in this case, and join AG.

Because BA is equal to BG (each being equal to DE).

Therefore, BAG is equal to BGA (Prop. 5, *Cor.*).

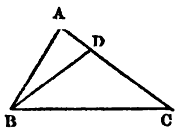
Similarly, CAG is equal to CGA.

Therefore, BAC is equal to BGC, i.e. to EDF, and similarly for the other angles.

PROPOSITION 8.

The greater angle of every triangle is subtended by the greater side, and the greater side of every triangle is subtended by the greater angle.

Fig. 26.



1st. Let ABC be a triangle having the angle ABC greater than the angle ACB then shall the side AC be greater than the side AB.

Because ABC is greater than ACB, a straight line BD may be drawn within the angle ABC making the angle DBC equal to the angle ACB; let this line meet AC in D.

Because DBC is equal to DCB, therefore the side DB is equal to the side DC (Prop. 6, *Cor.*).

To each of these equals add DA,

therefore BD+DA is equal to DC+DA.

But DC+DA is equal to AC, and BD+DA is greater than AB (Prop. 1),

therefore AC is greater than AB.

2nd. Let AC be greater than AB, then the angle ABC shall be greater than the angle ACB.

The angle ABC cannot be equal to the angle ACB, for then the side AB would be equal to the side AC (Prop. 6, *Cor.*), which it is not.

Also the angle ABC cannot be less than the angle ACB, for then the side AC would be less than the side AB (Part 1), which it is not.

Since, therefore, the angle ABC is neither equal to, nor less than, the angle ACB, it must be greater than it.

PROPOSITION 9.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle contained by the two sides of the one greater than the angle contained by the two sides equal to them of the other, the base of that which has the greater angle shall be greater than the base of the other.

Fig. 27 *a*, *b*, *c*.

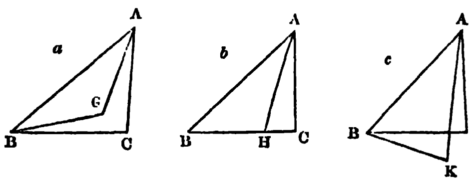
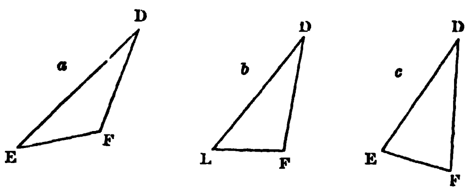


Fig. 28 *a*, *b*, *c*.



Let ABC and DEF be two triangles having the sides BA, and AC of the one equal, respectively, to the sides ED and DF of the other, but the angle BAC greater than the angle EDF; then shall the side BC be greater than the side EF.

Apply the triangle DEF to the triangle ABC, so that the

point D may be on A, and the straight line DE on AB, and let both triangles lie on the same side of AB.

Because the point D coincides with the point A, and DE coincides with, and is equal to, AB, therefore the point E coincides with the point B.

Because the angle EDF is less than the angle BAC, the straight line DF must coincide with some line lying between AB and AC.

Therefore the point F must coincide with some point G within the triangle ABC, or with some point H upon BC, or with some point K without the triangle ABC, such that the line AK cuts the side BC.

If F coincides with G, then

Because AC+CB is greater than AG+GB (Prop. 2),
and AC is equal to AG, each being equal to DF,
therefore BC is greater than BG,
therefore BC is greater than EF.

If F coincides with H, then

BC is greater than BH,
therefore BC is greater than EF.

If F coincides with K, then

BC+AK is greater than AC+BK (Prop. 3),
also AK is equal to AC, each being equal to DF,
therefore BC is greater than BK,
therefore BC is greater than EF.

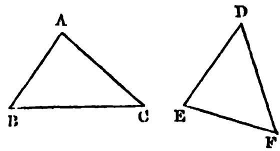
PROPOSITION 10.

If two triangles have two sides of one of them, respectively, equal to two sides of the other, but the base of the one greater than the base of the other, the angle contained by the two sides of the one shall be greater than the angle contained by the two sides equal to them of the other.

Let ABC and DEF be two triangles having the sides AB and AC, respectively, equal to the sides DE and DF,

but the base BC greater than the base EF, then shall the angle BAC be greater than the angle EDF.

Fig. 29.



The angle BAC cannot be equal to the angle EDF, for then the base BC would be equal to the base EF (Prop. 5), which it is not.

Again, the angle BAC cannot be less than the angle EDF, for then the base BC would be less than the base EF (Prop. 9), which it is not, therefore the angle BAC must be greater than the angle EDF.

EXAMPLES.

1. If a point be taken within a triangle, and lines be drawn from it to the angular points of the triangle, prove that these straight lines will be together less than the sum of the sides of the triangle,* and greater than half their sum.

2. If a straight line be drawn from the vertex of a triangle to the middle point of the opposite side, prove that this line will be less than half the sum of the sides which meet in the vertex, and greater than half the difference of this sum and the third side.

3. If lines be drawn (as in the last example) from each of the vertices of the triangle in succession, prove that the perimeter of the triangle will be greater than the sum of these lines, and less than double that sum.

4. ABC is a triangle; on AB, produced if necessary, take AD equal to AC, and on AC take AE equal to AB and draw DE meeting BC in F. Prove that AF bisects the angle BAC.

5. ABC is an isosceles triangle, of which the side AB is equal to AC, and D is the middle point of BC: in AC a

* This sum is called the perimeter of the triangle.

point E is taken; prove that the difference of DB and DE is less than the difference of AB and AE.

6. Prove that two triangles are equal in all respects when they have two sides equal, each to each, and likewise the lines joining the middle points of these sides with the opposite angles equal in each.

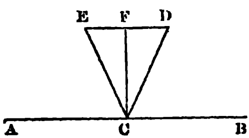
7. ADB is a triangle, of which the angle B is greater than the angle A, and C is a point on the same side of AB as the triangle, such that CA and CB are each equal to half the sum of DA and DB. If CB cuts AD in E, prove that EA is greater than EB, and EC greater than ED.

SECTION II.—ON ANGLES.

PROPOSITION 11.

From a given point in a given straight line, it is always possible to draw a straight line at right angles to the given straight line. And every straight line drawn from the given point on the same side of the given straight line makes the adjacent angles together equal to two right angles.

Fig. 30.



Let C be the given point, and AB the given straight line. Draw any finite straight line CD, from the point C above AB, and so that the angle BCD is less than the adjacent angle ACD.

Let the angle BCD be transferred so that the arm CB may coincide with CA, and let CE be the line with which the arm CD then coincides, D falling on E.

Join ED, bisect ED in F, and join CF, then CF shall be at right angles to AB.

Because CD coincides with CE therefore CD is equal to CE, also the angle ACE coincides with and is therefore equal to the angle ACE.

Because in the two triangles ECF and DCF, the sides

CE, EF, and FC are respectively equal to the sides CD, DF, and FC, the two triangles are equal in all their parts.

Therefore the angle DCF is equal to the angle ECF, and the angle DFC to the angle EFC.

Because the angle FCE is equal to the angle FCD, and the angle ECA is equal to the angle DCB,

therefore the angle FCA is equal to the angle FCB,

therefore FC is at right angles to AB.

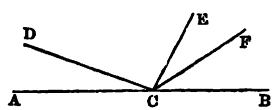
Again, because the angle DCB falls short of the right angle FCB by the angle FCD, and the angle DCA exceeds the right angle FCA by the same angle FCD, therefore DCA + DCB is equal to the sum of the two right angles FCA and FCB,

or DCA + DCB is equal to two right angles.

Corollary 1.—If a line be drawn from the vertex of an isosceles triangle to the middle point of the base, this line will bisect the vertical angle and will also be perpendicular to the base, for EFC and DFC are right angles, being equal adjacent angles.

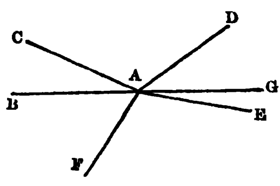
Corollary 2.—If any number of straight lines, as CD, CE, and CF, be drawn from a point C, in the straight line AB, and on the same side of it, then the sum of the angles ACD, DCE, ECF, and FCB, will be together equal to *two right angles.*

Fig. 31.



Corollary 3.—If any number of straight lines, as AB, AC, AD, AE, and AF, all meet in the point A, then the sum of the angles BAC, CAD, DAE, EAF, and FAB, will be together equal to *four right angles.* For if BA be produced to G, it is evident that the sum of these angles is equal

Fig. 32.



to the sum of the angles BAC, CAD, and DAG, together with the sum of the angles GAE, EAF, and FAB, and each of these sums is equal to two right angles by the last corollary.

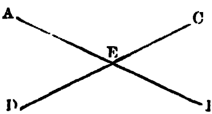
DEFINITIONS.

21.—An angle less than a right angle is called an *acute* angle, and an angle greater than a right angle is called an *obtuse* angle.

22.—The angle by which any given angle falls short of two right angles is called the *supplement* of the given angle.

23.—The angle by which any given angle falls short of one right angle is called the *complement* of the given angle.

Fig

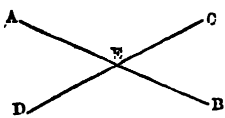


24.—If two straight lines, as AB and CD, cut each other in the point E, the angles AEC and BED are said to be *vertically opposite* angles, and so are the angles AED and CEB (Fig. 33).

PROPOSITION 12.

If two straight lines cut one another, the vertically opposite angles shall be equal.

Fig. 34.



Let the straight lines AB and CD cut one another in the point E, then the vertically opposite angles AEC and BED shall be equal to each other, as also the angles AED and BEC.

Because AE stands upon CD,
therefore AEC + AED is equal to two right angles (Prop. 11).

And because CE stands upon AB,
therefore AEC + CEB is equal to two right angles (Prop. 11),
therefore AEC + AED is equal to AEC + CEB.

Take away the common angle AEC,

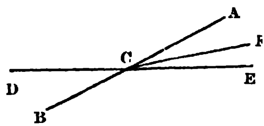
therefore the angle AED is equal to the angle BEC.

Similarly it may be proved that the angle AEC is equal to the angle BED.

PROPOSITION 13.

If at a point in a straight line two straight lines on opposite sides of it make the adjacent angles together equal to two right angles, or make the vertically opposite angles equal to one another, these two straight lines shall be in one and the same straight line.

Fig. 35.



1st. At the point C in the straight line AB, and on opposite sides of AB, let the two straight lines DC and CE make the adjacent angles ACD and ACE together equal to two right angles, then shall DC and CE be in the same straight line.

If CE be not in the same straight line with DC let, if possible, CF be in the same straight line with DC.

Because CA is drawn from the point C in the straight line DCF

therefore DCA + ACF is equal to two right angles (Prop. 11).

But, by hypothesis,

DCA + ACE is equal to two right angles,

therefore DCA + ACF is equal to DCA + ACE.

Take away the common angle DCA,

therefore the angle ACE is equal to the angle ACF which is impossible.

Therefore CF is not in the same straight line with DC, and similarly it may be proved that no other line, except CE, is in the same straight line with DC,

therefore DCE is a straight line.

2nd. Let the two straight lines DC and CE on opposite sides of AB make the vertically opposite angles DCB and

ACE equal, then shall DC and CE be in the same straight line.

If not, let CF be in the same straight line with DC.

Because the straight lines ACB and DCF intersect in C, therefore the angle DCB is equal to the angle ACF (Prop. 12).

But, by hypothesis,

the angle DCB is equal to the angle ACE.

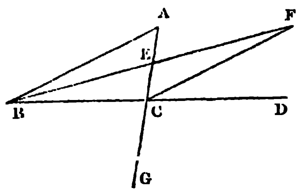
Therefore the angle ACE is equal to the angle ACF, which is impossible.

Therefore CF is not in the same straight line with DC, and similarly it may be proved that no other line than CE can be in the same straight line with DC, therefore DC and CE are in the same straight line.

PROPOSITION 14.

The exterior angle, formed by producing any one side of a triangle, is greater than either of the interior and non-adjacent angles, and any two angles of a triangle are together less than two right angles.

Fig. 36.



Let ABC be a triangle, and let the side BC be produced to D, then

1st. The exterior angle ACD shall be greater than either of the interior non-adjacent angles BAC or ABC.

Let E be the middle point of AC, join BE and produce it to F, making EF equal to BE, and join CF.

Because the two sides AE and EB, and the angle AEB, are respectively equal to the two sides CE and EF, and the angle CEF (Prop. 12 and construction),

therefore the triangles AEB and CEF are equal in all their parts,

therefore the angle ACF is equal to the angle BAC,

therefore the angle ACD is greater than the angle BAC.

If AC be produced to G, it may be proved in a similar manner that the angle BCG is greater than the angle ABC.

But the angle BCG is equal to the angle ACD (Prop. 12),
therefore the angle ACD is greater than either of the angles BAC or ABC.

2nd. Any two angles of the triangle, as ABC and ACB, shall be together less than two right angles.

Because the angle ACD is greater than the angle ABC,
therefore ACD + ACB is greater than ABC + ACB.
But ACD + ACB is equal to two right angles (Prop. 11),
therefore ABC + ACB is less than two right angles.

PROPOSITION 15.

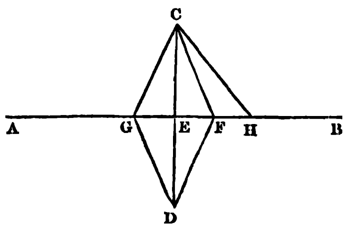
If a point be taken outside a given straight line, then—

1. *It shall be always possible to draw one line, and only one line, from this point perpendicular to the given line, and lines equally inclined to this perpendicular shall be equal to one another.*

2. *The perpendicular shall be the shortest line that can be drawn from the given point to the given line, and of any pair of lines unequally inclined to the perpendicular the line nearer to it shall be less than that more remote.*

Let AB be the given straight line, and C the point without it.

Fig. 37.



1st. It shall be always possible to draw one, and only one, line from C perpendicular to AB, and lines to AB from C equally inclined to this perpendicular shall be equal to one another.

Suppose the plane which contains the point C and the line AB to be turned about the line AB until the portion of the plane above AB is brought into coincidence with the portion of the same plane below

AB, and let D be the point with which the point C coincides in this case.

Join CD, meeting AB in E, take F, any other point in AB, and join CF and DF.

Because the straight line EC is made to coincide with the straight line ED, while AB remains fixed,

therefore the angle AEC is equal to the angle AED,

therefore each of the angles AEC and AED is a right angle.

But CEB is equal to AED (Prop. 12), therefore CEB and AEC are each right angles, and EC is perpendicular to AB.

Also no other line except CE can be drawn from C at right angles to AB.

For, if it were so drawn, we should have two angles of a triangle together equal to two right angles, which is impossible, by Prop. 14.

Let CG be a line from C, such that the angle ECG is equal to the angle ECF.

Then in the triangles ECG and ECF the two angles FEC and FCE of the one are equal to the two angles GEC and GCE of the other, each to each, and the side EC adjacent to the equal angles is common;

therefore the triangles are equal in all their parts, and the side CG is equal to the side CF.

2nd. CE is the shortest line from C to AB, and CF which is nearer to CE, is less than CH, which is more remote.

Because CEF is a right angle, therefore CFE is less than a right angle (Prop. 14),

therefore CFE is less than CEF,

therefore CE is less than CF (Prop. 8),

i.e. CE is less than any other straight line from C to AB.

For the same reason as above the angle CHF is less than a right angle.

Also the angle CFH is greater than the interior angle CEF, and therefore greater than a right angle (Prop. 14),

therefore CFH is greater than CHF,

therefore CF is less than CH (Prop. 8).

Corollary.—If one straight line bisect another straight line at right angles, then every point of the bisecting line will be equidistant from the extremities of the bisected line.

EXAMPLES.

1. If two straight lines cut one another, prove that the two bisectors of the angles formed at their point of intersection are perpendicular to each other.

2. Two straight lines, CA and CB, intersect in a point C, and CO is a line through C bisecting the angle ACB. Prove that if CD be any other line whatever drawn through C, the angle DCO will be equal to half the sum or half the difference of the angles DCA and DCB, according as CD is without or within the angle ACB.

3. Prove that the perpendiculars drawn from the extremities of the base of an isosceles triangle upon the opposite sides are equal to one another.

4. ABC is a triangle, and AD the line bisecting the angle BAC; through A the line EAF is drawn at right angles to AD. Prove that if P be any point on the line EF, the perimeter of the triangle BPC will be greater than that of ABC.

5. If two straight lines intersect in a point, and one of the four angles be a right angle, prove that all the rest are right angles.

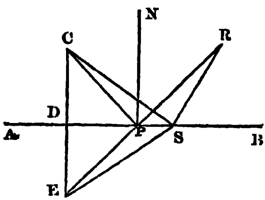
6. Two angles are supplementary, and the greater is double of the less; find what fraction the less is of two right angles.

7. From a given point in a given straight line, two straight lines, and only two, can be drawn, so that each may make a given angle with the given line.

Note.—Suppose the plane of the paper to be that of a billiard table, and AB to be one of the cushions. Let a ball be struck so as to impinge upon the cushion in the direction CP, and let PN be at right angles to AB. Then P is called the *point of incidence*, and CPN the *angle of incidence*, and it

is proved in Mechanics that if PR be drawn through P, making the angle RPN equal to CPN, the ball will, if its elasticity be perfect, after striking the cushion, rebound in the direction PR; the line PR is called the line of reflection, and the angle NPR the angle of reflection, and the property alluded to is generally stated thus: *the angle of reflection is equal to the angle of incidence.*

Fig. 38.



Draw CD perpendicular to AB and produce it to E, making DE equal to CD and join PE, then it follows from the text that DPE and DPC are equal angles. But RPB is equal to DPC, therefore RPB is equal to DPE, therefore EPR is a straight line, and therefore whatever be the point of incidence the ball rebounds as if it came from E.

Or again, if the plane of the paper were some horizontal plane, and AB the section by this plane of a vertical plane reflecting mirror, and C a luminous point, then it is proved in optics that if a ray of light were incident upon AB from C in the direction CP, it would be reflected in a direction PR, such that the angles of incidence and reflection were equal. Therefore *every* ray after reflection proceeds as if it came from E, wherever may be the point of incidence, and if an eye were situated at R it would look towards E to receive the reflected light. Therefore the effect to observers is in all respects the same as if a luminous spot were situated at E. The point E is called the *image* of the point C.

If S be a point in AB different from P, and the lines CS, ES and RS be joined, we have

CP equal to EP and CS equal to ES;

therefore CP + PR is equal to EP + PR, i.e. equal to ER. But ER is less than ES + SR, i.e. less than CS + SR.

Or, in other words, If two given points (C and R) be situated on the same side of a given straight line, and a

point be found in this line, such that the lines joining it with the given points make equal angles with the opposite parts of the line, these two lines will be together less than the sum of the two lines drawn from the given points to any other point in the given line.

PROPOSITION 16.

If two right-angled triangles have their hypothenuses equal, and a side of the one equal to a side of the other, the two triangles shall be equal to one another in all their parts.

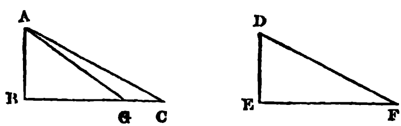
Let ABC and DEF be two right-angled triangles, having the angles at B and E right angles, and the hypothenuses AC and DF equal, also let the side AB be equal to the side DE; then shall the triangles ABC and DEF be equal to one another in all their parts.

Let the triangle DEF be applied to the triangle ABC so that the point E may fall on the point B, and the side ED on the side BA, and let the triangles lie on the same side of AB.

Because the point E coincides with the point B and the straight line ED coincides with and is equal to the straight line BA, therefore the point D coincides with the point A.

Because ED coincides with BA, and the angle DEF is equal to the angle ABC, therefore EF coincides with BC;

Fig. 39.



therefore DF must coincide with AC, for, if it do not, let it take some other direction as AG, within the angle BAC.

Because AB is the perpendicular from A upon BC, and AG is nearer to AB than AC, therefore AG is less than AC (Prop. 15).

But AG is also equal to AC, because each of them is equal to DF,

which is impossible;

therefore DF cannot fall within the angle BAC, and in like manner it may be proved that DF cannot fall without the angle BAC, therefore DF must coincide with AC,

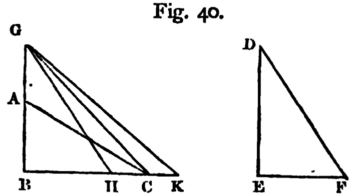
therefore the points A, B, and C coincide with the points D, E, and F respectively,

and therefore the triangles DEF and ABC are equal in all their parts (Prop. 5, *Cor.* 2).

PROPOSITION 17.

If two right-angled triangles have their hypothenuses equal, and one side of the one greater than one side of the other, then shall the remaining side of the one be less than the remaining side of the other.

Fig. 40.



Let ABC and DEF be two right-angled triangles, having the angles at B and E right angles, and the hypothenuses AC and DF equal. Also let the side DE be greater than the side AB, then shall the side EF be less than the side BC.

Apply the triangle DEF to the triangle ABC so that the point E coincides with the point B, and the side ED with the side BA, and let the triangles lie on the same side of AB.

Because E coincides with B and ED coincides with and is greater than BA, therefore the point D coincides with some point G in BA produced.

Because ED coincides with BA, and the angle DEF is equal to the angle ABC, therefore EF coincides with BC.

Then DF must coincide with some line as GH, meeting BC between B and C.

For, if not, it must either coincide with GC, or some such line as GK without the triangle ABC.

Because CB is perpendicular to GB, and the angle BCA less than the angle BCG, therefore CA is less than CG (Prop. 15).

Similarly because GB is perpendicular to BC, therefore GC is less than GK,

therefore AC is less than either GC or GK.

But DF is equal to AC, and therefore DF cannot coincide either with GC or GK :

therefore DF must coincide with some such line as GH, but BH is less than BC,

therefore EF is less than BC.

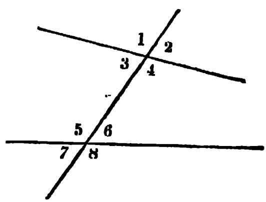
Section III.—On Parallel Straight Lines.

DEFINITIONS.

When a straight line intersects two other straight lines it makes with them four angles at each of the two points of intersection, which angles have received special names.

25.—In the figure 1, 2, 7, 8 are called *exterior* angles; and 3, 4, 5, 6 are called *interior* angles.

Fig. 41.



26.—Again, 1 and 5 are said to be *corresponding* angles; so also are 2 and 6, 7 and 3, 8 and 4; and 3 and 6 are called *alternate* angles; so also are 4 and 5.

27.—Two straight lines which are situated in the same plane, and cannot meet, however far they may be produced, are called *parallel* lines.

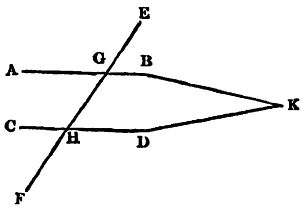
Axiom 4.—If two straight lines be each of them parallel to a third straight line they cannot meet one another; or, in other words, through any one point no more than one straight line can be drawn parallel to the same straight line

PROPOSITION 18.

If a straight line, cutting two other straight lines, make the angles in either pair of alternate angles, or in any pair of corresponding angles equal to each other, or make the two interior angles on either side of the cutting line together equal to two right angles, these two straight lines shall be parallel.

1st. Let the straight line EF, cutting the two straight lines AB and CD, make the angles in either pair of alternate angles, as AGH and GHD equal to each other, then AB shall be parallel to CD.

Fig. 42.



For, otherwise, AB and CD will meet when produced far enough either towards A and C or towards B and D.

Let them meet, when produced towards B and D, in the point K.

Because KGH is a triangle, and the side KG is produced to A, therefore the exterior angle AGH is greater than the interior angle GHD.

But, by hypothesis, the angle AGH is equal to the angle GHD,

which is impossible,

therefore AB and CD do not meet when produced towards B and D.

Similarly, it may be proved that they do not meet when produced towards A and C,

therefore AB and CD are parallel.

2nd. Let the angles in any pair of corresponding angles, as EGB and GHD, be equal to each other, then AB shall be parallel to CD.

Because the angle EGB is equal to the angle AGH (Prop. 12),
and the angle EGB is equal to the angle GHD by hypothesis,
therefore the angle AGH is equal to the angle GHD,
therefore AB and CD are parallel by the first case.

3rd. Let the two interior angles BGH, and GHD, on the same side of GH, be together equal to two right angles, then shall AB and CD be parallel.

Because HG stands on AB,
therefore AGH + BGH is equal to two right angles (Prop. 11),
therefore AGH + BGH is equal to BGH + GHD;
take away the common angle BGH;
therefore the angle AGH is equal to the angle GHD,
therefore AB is parallel to CD by the first case.

Corollary.—If a straight line cut two other straight lines, and either of the two *acute*, or either of the two *obtuse* angles formed at the point of intersection with the one line be equal to either of the two *acute*, or either of the two *obtuse* angles, formed at the point of intersection with the other, then these two straight lines shall be parallel.

PROPOSITION 19.

If a straight line cut two parallel straight lines, it shall make the angles in either pair of alternate angles, or in any pair of corresponding angles, equal to each other, and also the two interior angles on either side of the cutting line together equal to two right angles.

Let the straight line EF cut the two parallel straight lines AB and CD, then—

1st. The angles in either pair of alternate angles as AGH and GHD shall be equal to each other.

If AGH be not equal to GHD we may draw through

H a line KHL, so that AGH may be equal to GHL; and this line will be situated as in Fig. 43 or Fig. 44, according as AGH is greater or less than GHD,

therefore AB is parallel to KL by the last proposition.

Fig. 43.

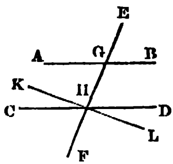
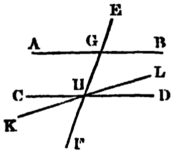


Fig. 44.



But AB is parallel to CD,

therefore two straight lines KL and CD are drawn through the same point H, each of them parallel to AB, which is impossible (*Ax.* 4);

therefore the angle AGH is equal to the angle GHD.

2nd. The angles in any pair of corresponding angles, as EGB and GHD, shall be equal to each other.

Because AB is parallel to CD, therefore the angles AGH and GHD are equal by the first case.

Because EGB and AGH are vertically opposite angles, therefore the angle AGH is equal to the angle EGB (Prop. 12),

therefore the angle EGB is equal to the angle GHD.

3rd. The two interior angles on either side of EF as BGH and GHD, shall be together equal to two right angles.

Because AB and CD are parallel, therefore the angle EGB is equal to the angle GHD by the second case,

therefore EGB+BGH is equal to BGH+GHD.

But EGB+BGH is equal to two right angles, therefore BGH+GHD is equal to two right angles.

Corollary 1.—If two straight lines are parallel they are equally inclined to any straight line which cuts them both, i.e. each acute angle made by the cutting line with either of the two parallels is equal to each acute angle made by

the cutting line with the other of the two parallels, and similarly for the obtuse angles.

Corollary 2.—If two straight lines are each perpendicular to the same straight line they are parallel to one another, and, conversely, if one of two parallel lines be perpendicular to a given straight line the other of the two parallel lines will be also perpendicular to that straight line.

PROPOSITION 20.

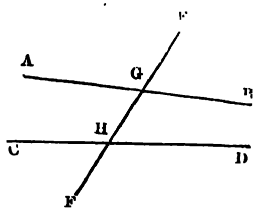
If a straight line cut two other straight lines, and make the two interior angles on either side of the cutting line, together, less than two right angles, these two straight lines, if produced far enough, shall meet on that side of the cutting line on which this pair of interior angles is situated.

Let the straight line EF, cutting the two straight lines AB and CD make the two interior angles BGH and GHD, together, less than two right angles.

AB and CD, if produced far enough towards B and D, shall meet.

If AB and CD do not meet when produced towards B and D, they must either be parallel or else meet when produced towards A and C.

Fig. 45.



If AB and CD be parallel, the angles BGH and GHD will be, together, equal to two right angles.

But BGH and GHD are together less than two right angles, therefore AB and CD are not parallel.

If AB and CD meet when produced towards A and C, the angles AGH and CHG will be two of the angles of a triangle, and therefore together less than two right angles (Prop. 14).

But the angles BGH and GHD are also, together, less than two right angles,

therefore the four angles AGH, BGH, CHG, and DHG, are together less than four right angles.

But the angles in each pair of angles AGH and BGH, CHG and DHG, are together equal to two right angles, therefore the four angles AGH, BGH, CHG, and DHG, are together *equal* to four right angles, and we have already shown that these four angles are together *less than* four right angles, which is impossible;

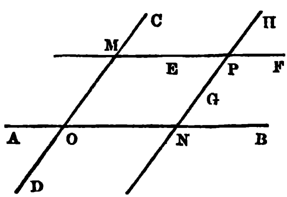
therefore AB and CD do not meet towards A and C, and it has been proved that they are not parallel, therefore AB and CD meet towards B and D.

PROPOSITION 21.

If two intersecting straight lines be parallel to two other intersecting straight lines each to each, then each of the acute angles at one of the points of intersection shall be equal to each of the acute angles at the other point of intersection, and similarly for the obtuse angles.

Let the two straight lines AB and CD intersecting at O be respectively parallel to the two straight lines EF and GH intersecting at P, then shall each of the acute angles at O be equal to each of the acute angles at P, and each of the obtuse angles at O to each of the obtuse angles at P.

Fig. 46.



Because OC meets OB, and PE is parallel to OB, therefore PE or PE produced must meet OC or OC produced in some point M (*Ax.* 4).

Because OC meets the parallels MP and OB in the

points M and O, therefore each of the acute angles at M is equal to each of the acute angles at O (Prop. 18, *Cor.* 1).

Similarly each of the acute angles at M is equal to each of the acute angles at P,

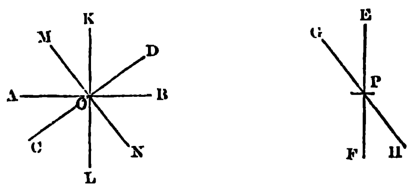
therefore each of the acute angles at O is equal to each of the acute angles at P.

Similarly each of the obtuse angles at O may be proved to be equal to each of the obtuse angles at P.

PROPOSITION 22.

If two intersecting straight lines be perpendicular to two other intersecting straight lines each to each, then each of the acute angles at one of the points of intersection shall be equal to each of the acute angles at the other point of intersection, and similarly for the obtuse angles.

Fig. 47.



Let the straight lines AOB and COD intersecting at O be perpendicular to the straight lines EPF and GPH intersecting at P, each to each, then shall each of the acute angles at O be equal to each of the acute angles at P, and each of the obtuse angles at O to each of the obtuse angles at P.

Let the figure formed by the two intersecting straight lines AOB and COD be turned round O, until AOB coincides with KOL at right angles to AOB, and let COD coincide with MON in this case.

Because the line AOB has been turned through a right angle,

therefore the line COD has been turned through a right angle.

Because KL and EF are each perpendicular to AOB, therefore they are parallel (Prop. 19, *Cor.* 2).

Similarly MN and GH are parallel,

therefore each of the acute angles between KL and MN at O is equal to each of the acute angles between EF and GH at P (Prop. 21).

Because AB and CD have been made to coincide with KL and MN respectively, therefore each of the acute angles between KL and MN at O is equal to each of the acute angles between AB and CD at O,

therefore each of the acute angles between AB and CD at O is equal to each of the acute angles between EF and GH at P.

Similarly each of the obtuse angles between AB and CD at O is equal to each of the obtuse angles between EF and GH at P.

EXAMPLES.

1. If through the middle point D of the side AB of the triangle ABC a line be drawn parallel to BC, prove that the latter will pass through E, the middle point of AC, and that DE will be equal to half of BC.

2. If AC and BD be two parallel straight lines, and ABO CDO two lines meeting in O, such that the intercepts AB and CD are equal, prove that AO will be equal to CO, and BO to DO.

3. If the lines containing one of two angles be parallel respectively to the lines containing the other, prove that the bisectors of these angles will be either perpendicular or parallel to one another.

4. The angle between the bisector of the angle A of the triangle ABC and the perpendicular from A on BC is equal to half the difference of B and C.

5. If the lines containing one of two angles be perpen-

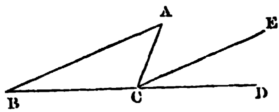
dicular respectively to the lines containing the other, prove that the bisectors of these angles will be either perpendicular or parallel to one another.

PROPOSITION 23.

If one side of a triangle be produced, the exterior angle shall be equal to the two interior non-adjacent angles, and the three angles of every triangle are together equal to two right angles.

1st. Let the side BC of the triangle ABC be produced to D, then shall the exterior angle ACD be equal to the two interior non-adjacent angles ABC and BAC.

Fig. 48.



Through C draw CE parallel to AB.

Because AC cuts the two parallel lines AB and CE

therefore the alternate angles BAC and ACE are equal to each other (Prop. 19).

Because BCD cuts the two parallel lines AB and CE

therefore the corresponding angles ABC and ECD are equal to each other (Prop. 19.).

therefore BAC+ABC is equal to ACE+ECD, i.e. to ACD.

2nd. Because ABC+BAC is equal to ACD,

add ACB to each;

therefore ABC+BAC+ACB is equal to ACD+ACB.

But ACD+ACB is equal to two right angles (Prop. 11);

therefore ABC+BAC+ACB is equal to two right angles.

Corollary.—If two angles of one triangle be equal to two angles of another triangle, the third angle of the first triangle will be equal to the third angle of the second.

Note.—We have seen in Props. 5, 6, and 7, that two triangles are equal in all their parts when—

1st. Two sides and the included angle of the one are respectively equal to two sides and the included angle of the other; or

2nd. When two angles and the included side of the one are respectively equal to two angles and the included side of the other; or

3rd. When the three sides of the one are respectively equal to the three sides of the other.

From the Corollary to Prop. 23 it follows that case 2 may be extended to the case of two angles, and *any* side of the one being respectively equal to two angles and a side of the other, provided the equal sides in each triangle be *included between* or *opposite* to equal angles.

And therefore we may say generally that two triangles are equal in all their parts when—

1st. Two sides and the angle between them are equal in each;

2nd. Two angles and a side of one triangle are equal respectively to two angles and a side of the other, the sides which are equal in the two triangles being either included between or opposite to angles which are equal in the two triangles.

3rd. All three sides are equal in each.

If two sides and an angle of one triangle be respectively equal to two sides and an angle of the other, the equal angles being *opposite* to equal sides in the two triangles, the triangles will be equal in all their parts, provided the angles opposite to the other equal sides in each be either both acute or both obtuse, as we shall see hereafter (page 103).

SECTION IV.—ON POLYGONS.

DEFINITIONS.

28.—A polygon is a plane figure bounded by straight lines. These bounding lines are the *sides* of the polygon, and the points in which successive pairs of sides intersect are the *angular points* or *vertices* of the polygon, also any straight line joining two non-consecutive vertices is a *diagonal* of the polygon.

Thus ABCDE, Fig. 49, and ABCDE, Fig. 50, are polygons. Any line as AC or AD joining two non-consecutive

Fig. 49.

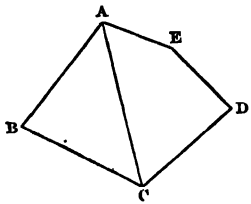
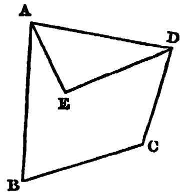


Fig. 50.



vertices as A and C in Fig. 49, or A and D in Fig. 50, is a diagonal.

29.—A polygon is termed *convex* when it lies entirely on the same side of every one of the sides of the polygon; thus ABCDE, Fig. 49, is a convex polygon, but ABCDE, Fig. 50, is not a convex polygon, because parts of the figure lie on opposite sides of one of its sides, viz. DE.

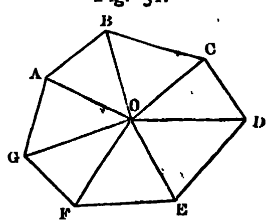
PROPOSITION 24.

The sum of the interior angles in any convex polygon is less than twice as many right angles as the figure has sides by four right angles.

Let ABCDEFG be any convex polygon.

Take O, any point within it, and join O with the angles of the polygon A, B, C, &c.

Fig. 51.



Then these lines divide the polygon into as many triangles as the figure has sides.

Therefore the sum of the angles in all these triangles is equal to twice as many right angles as the figure has sides (Prop. 23).

But this sum is the sum of the angles of the polygon ABC, BCD, &c., together with the angles formed at O.

Also the angles at O are together equal to four right angles,

Therefore the sum of the angles of the polygon together with four right angles is equal to twice as many right angles as the figure has sides, or the sum of the angles of the polygon is less than twice as many right angles as the figure has sides by four right angles.

Corollary.—If n be the number of sides, then the sum of the angles is equal to

$2n$ right angles less by 4 right angles,

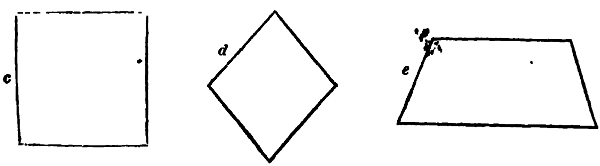
i.e. in algebraical language to $(2n-4)$ right angles.

DEFINITIONS.

30.—A *parallelogram* is a four-sided figure in which the opposite sides are parallel (a).

31.—A *rectangle* is a parallelogram in which one of the angles is a right angle (b).

32.—A *square* is a rectangle in which all the sides are equal (c).



33.—A *rhombus* is a four-sided figure in which all the sides are equal (d).

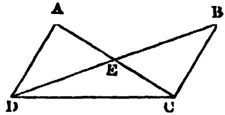
34.—A *trapezoid* is a four-sided figure in which two opposite sides are parallel to each other (e).

PROPOSITION 25.

In every parallelogram the opposite sides and angles are equal to one another respectively, and the diagonals bisect each other.

Let ABCD be a parallelogram, of which AC and BD are the diagonals intersecting in E, then the opposite sides and angles of ABCD shall be equal to one another respectively, and each of the diagonals AC and BD shall be bisected in E.

Fig. 52.



Because AC meets the parallel straight lines BC and AD, therefore the alternate angles DAC and ACB are equal.

Similarly, the angles BAC and DCA are equal

Therefore in the two triangles CAD and ACB the angles at A and C in the one are equal to the angles at C and A in the other, and the side AC between the equal angles in each is common ;

therefore the triangles CAD and ACB are equal in all their parts ;

therefore AB is equal to CD, and AD to CB, and the angle ADC to the angle CBA.

Also the parts CAD and CAB of the angle of the parallelogram at A have been proved to be respectively equal to the parts ACB and ACD of the angle of the parallelogram at C,

therefore the angle DAB is equal to the angle BCD.

Again : Because the angles EAB and EBA have been proved to be respectively equal to the angles ECD and EDC, and the side AB to be equal to the side CD,

therefore the triangles EAB and ECD are equal in all their parts ;

therefore AE is equal to EC and DE to EB.

Corollary 1.—Since the sum of the angles of any quadrilateral figure is equal to four right angles by Prop. 24, and the opposite angles of a parallelogram are equal, it follows that if one angle be a right angle all the angles will be right angles, or that all the angles of a rectangle are right angles.

Corollary 2.—If two triangles can be applied to one another so that their angles coincide each with each, they are equal to one another in all their parts, and if they be equal to one another in all their parts they may be so applied.

Examples.

1. From the extremities of the base of an isoceles triangle straight lines are drawn perpendicular to the opposite sides. Prove that the angles they make with the base are each equal to half the vertical angle.

2. Prove that a convex polygon cannot have more than three acute interior angles.

3. Prove that an angle of a triangle is equal to, less, or greater than a right angle according as the line drawn from it to the middle point of the opposite side is equal to, greater, or less than half that opposite side.

4. In any triangle ABC two lines AD and AE are drawn to BC, making with AB and AC angles equal to C and B respectively. Prove that DAE is an isosceles triangle.

5. If O be any point within the triangle ABC, prove that the angle BOC is always greater than the angle BAC.

6. In every triangle prove that the lines in each of the following systems of three lines meet in a point:

   i. The three perpendiculars from the angles upon the opposite sides.

   ii. The three bisectors of the angles.

   iii. The three lines from the angles to the middle points of the opposite sides.

Prove also that the bisectors of each interior angle and the two non-adjacent exterior angles meet in a point.

7. In a right-angled triangle if one of the acute angles be double of the other, the hypothenuse will be double the smaller side.

8. Every parallelogram whose diagonals are equal is a rectangle, every parallelogram whose diagonals are perpendicular to each other is a rhombus, and one whose diagonals are both equal and perpendicular to each other is a square.

9. ABCD is a parallelogram, and E and F are the middle points of the sides AD and BC respectively. Prove that BE and DF trisect the diagonal AC.

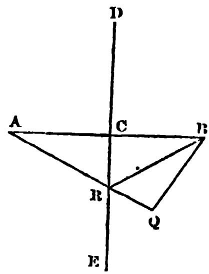
10. Prove that the four straight lines which join the middle points of adjacent sides of a quadrilateral form a parallelogram.

Section V.—On Loci.

Note.—We have seen (Prop. 15, *Cor.*) that every point in the straight line which bisects at right angles the straight line joining two given points is equidistant from these points. We may also prove that if a point be not situated in this line it cannot be equidistant from these points.

For let DCE be the line bisecting AB at right angles, and let Q be a point not lying in the line DE.

Fig. 53.



Join QA and QB, let AQ meet CE in R, and join RB.

RA is equal to RB, therefore AQ is equal to BR + RQ.

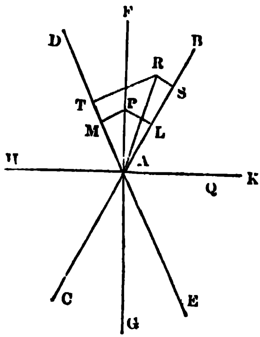
But BR + RQ is greater than BQ, and therefore AQ is greater than BQ, or the point Q is not equidistant from A and B.

When a line can be found, as in this case, such that every point in it satisfies a certain proposed condition, and that no point which is not situated in the line can satisfy that condition, the line is called the *locus* of a point satisfying the proposed condition. Thus in this case we say that the locus of a point which is equidistant from two given points is the straight line which bisects at right angles the straight line joining the two given points.

Again: Let BAC and DAE be two straight lines intersecting in A (Fig. 54). Draw FAG bisecting the angle BAD,

take any point P in FAG and draw PL and PM perpendicular to AB and AD.

Fig. 54.



Because the two triangles PAL and PAM have the angles at A equal, and the right angles at L and M equal, and AP common, therefore they are equal in all their parts, and therefore PM is equal to PL, i.e. every point in AF is equidistant from the lines AB and AD.

Let R be a point within the angle BAD but not situated in the line AF, and from R draw RS and RT perpendicular to AB and AD, then RS cannot be equal to RT. For if so the right-angled triangles ARS and ART would have the hypothenuse AR common, and the sides RS and RT equal, and therefore they would be equal in all respects, and SAR would be equal to TAR, and therefore SAP would be greater than TAR, and therefore still greater than TAP, which is contrary to this supposition.

In the same way it may be shown that if a point Q be taken in AK it will be equidistant from the lines BC and DE, and that no point within the angle BAE, not situated in AK, can be equidistant from those lines. The same thing may be proved of every point in AF or AK *produced.*

Therefore any point lying on either one of the straight lines FG and HK satisfies the proposed condition of being equidistant from the two given lines BC and DE, and no point not lying either on FG or HK satisfies this condition.

In this case the two straight lines FG and HK, *taken together*, would be said to constitute the locus of a point satisfying the proposed condition.

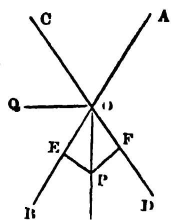
In solving questions concerning loci an answer to the

question as to the nature of the locus may generally be arrived at by *assuming the condition satisfied* and reasoning from the figure thus obtained. For example, in the latter of the two questions just considered we might proceed thus:

To find the locus of the point which is equidistant from two given intersecting straight lines.

Let AB and CD be the given lines intersecting in O. Assume a point P situated within the angle BOD to be a point on the locus, and from P draw PE and PF perpendicular to AB and CD, and join OP.

Fig. 55.



Because the right-angled triangles OPE and OPF have the hypothenuse OP common to both, and the side PE equal to the side PF, therefore they are equal to each other in all respects, and therefore the angles POE and POF are equal, therefore the line OP bisects the angle BOD, but only one line can be drawn through O bisecting the angle BOD, therefore the locus of P is that straight line.

Similarly, if we assumed a point Q within the angle COB to be equidistant from CD and AB we should find that Q must be situated on the line bisecting the angle COB, whence the result stated above as to the locus of a point equidistant from two given straight lines. Subjoined are a few additional examples of loci.

PROPOSITION 26.

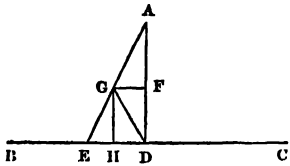
The locus of the middle points of the straight lines drawn to different points in a given straight line from a given point without it is the straight line bisecting at right angles the perpendicular from the given point upon the given straight line.

Let A be the given point and BC the given straight line.

From A draw AD perpendicular to BC and let AE be any straight line not perpendicular to BC and meeting BC in E.

Take G, the middle point of AE, and from G draw GF and GH perpendicular to AD and BD respectively, and join GD.

Fig. 56.



Because GH and AD are both perpendicular to BC therefore they are parallel to one another,

therefore the angle EGH is equal to the angle GAF (Prop. 19).

Therefore the angles EGH and GHE and the side GE of the triangle EGH are respectively equal to the angles GAF and GFA, and the side AG of the triangle GAF,

therefore the triangles EGH and GAF are equal in all their parts,

therefore GH is equal to AF.

Because GF and BD are each of them perpendicular to AD,

therefore GF is parallel to BD (Prop. 19, *Cor.* 2).

Also GH is parallel to AD,

therefore the figure GFDH is a parallelogram,

therefore GH is equal to FD (Prop. 25);

but GH has been proved to be equal to AF,

therefore FD is equal to AF, or the point G is on the line bisecting AD at right angles.

PROPOSITION 27.

The locus of a point from which the sum of the perpendicular distances upon two given intersecting straight lines is equal to a given finite straight line, is the base of that isosceles triangle formed by the given lines, such that the perpendicular distance of either of the angles at the base from the opposite side is equal to the given finite straight line.

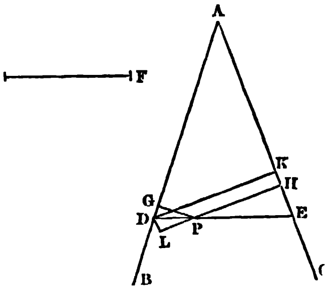
Let AB and AC be the two given straight lines intersecting in A, and F the given finite straight line.

Assume P to be a point on the locus between AB and

AC, and draw PG and PH perpendicular to AB and AC respectively.

Through P draw DPE, cutting off an isosceles triangle from BAC. Then shall the perpendicular distance of D from AC or of E from AB be equal to F.

Fig. 57.



Draw DK and DL parallel to PH and AC respectively, and let HP produced meet DL in the point L.

Then DH is a parallelogram and DK is equal to HL.

Because DL is parallel to AC and LPH is perpendicular to AC, therefore PL is perpendicular to DL.

Because DL is parallel to AC therefore PDL is equal to AED, and therefore also to ADP,

therefore in the triangles PDG and PDL the angles at D are equal, and the angles at G and L are equal, being right angles, and the side PD is common,

therefore PL is equal to PG,

therefore LH is equal to PG+PH, and therefore LH is equal to F,

therefore also DK is equal to F.

And it may be easily proved that if P be not in DE, the sum of the perpendiculars let fall from P upon AB and AC cannot be equal to F.

Examples.

1. Find the locus of a point which is always at the same given distance from a given straight line.

2. Find the locus of a point such that the *differences* of its distances from two given straight lines may be of constant length.

Miscellaneous Examples on Book I.

1. Prove that the angle between the bisectors of two consecutive angles of a quadrilateral is equal to half the sum of its other two angles.

2. Prove that the sum of the distances of any point taken within an equilateral triangle from the three sides is the same whereon the point may be. Investigate the case in which the point is without the triangle.

3. AD and BC are two parallel straight lines cut obliquely by AB and perpendicularly by AC, and the straight line BED is drawn between them, cutting AC in E, so that ED is equal to twice AB. Prove that the angle DBC is one third of ABC.

4. On the side AB of the triangle ABC take AD equal to AC, produce BA and take AE equal to AC, then join the angle C to the points D and E. Prove that the angle at E is half BAC and that DCE is a right angle.

5. Prove that the point of intersection of the straight lines joining the middle points of opposite sides of any quadrilateral is the middle of the straight line joining the middle points of the diagonals.

6. ABC is a triangle right-angled at A, and ABDM and ACEN are squares upon AB and AC respectively. From the points D and E perpendiculars DF and EG are drawn to the hypothenuse BC produced. Prove that the hypothenuse BC is equal to the sum of the perpendiculars DF and EG.

7. In every trapezoid the middle points of the two non-parallel sides and of the two diagonals are on one straight line which, parallel to the other two sides and the distance between the two extreme points, is equal to half the sum of these sides and the distance between the intermediate points to half their difference.

8. Prove that if a ball lying on a rectangular billiard

table be set in motion in a direction parallel to one of the diagonals of the table, it will, after rebounding from each cushion in succession, return to the point of departure, and that the path described by it will be equal to the sum of the diagonals of the table.

9. If on the sides of a square ABCD any four points E, F, G, and H be taken, one on each side, so that $AE = BF = CG = DH$, prove that EFGH will also be a square.

10. If through any point on the base of an isosceles triangle parallels be drawn to the other two sides, prove that the parallelogram so formed will have the same perimeter wherever the point may be situated.

BOOK II.

ON THE CIRCLE.

SECTION I.—MISCELLANEOUS PROPOSITIONS.

DEFINITIONS.

35.—The locus of a point which is always in one plane and at the same given distance from a given point in that plane is a *circle*; the given point is called the *centre*, and the given distance is called the *radius* of the circle.

36.—The portion of the circle between any two points upon it is called *an arc of the circle*, and the straight line joining the two points is called *the chord of the arc.*

N.B.—The line forming the locus is sometimes called the circumference of the circle and sometimes the circle.

PROPOSITION I.

A circle is a finite closed figure, such that every point in the plane of the circle whose distance from the centre is less than the radius of the circle is situated within the circle; and every point in the plane of the circle whose distance from the centre is greater than the radius of the circle is situated without the circle.

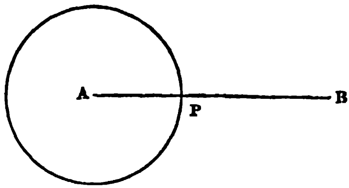
Let A be the centre of any given circle.

Draw the indefinite straight line AB terminated at A, but unlimited towards B.

Because AB is a straight line of indefinite length, therefore it is always possible to find a point P upon AB, such that AP is equal to the radius of the circle.

Also every point on AB between P and B is at a distance from A greater than the radius of the circle; and every point on AB between A and P is at a distance from A less than the radius of the circle.

Fig. 1.



Therefore every indefinite straight line drawn from A meets the circle in one point, and one point only.

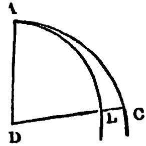
Therefore the circle is a finite closed figure, such that every point in the plane of the circle whose distance from A is less than the radius of the circle is situated within the circle; and every point in the plane of the circle whose distance from A is greater than the radius of the circle is situated without the circle.

PROPOSITION 2.

Two circles which have the same centre and one point in common must coincide in every point.

Let, if possible, AB and AC be non-coincident arcs of two circles having the same centre D and the point A common to both.

Fig. 2.



Draw any straight line through D, meeting these arcs in B and C, and join DA.

Because D is the centre of the circle of which AB is an arc, therefore DA is equal to DB.

Similarly, DA is equal to DC,

therefore DB is equal to DC,

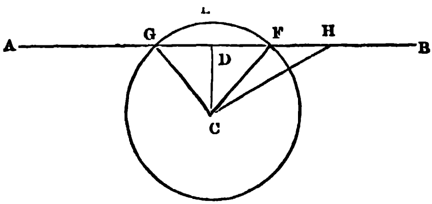
which is impossible.

PROPOSITION 3.

If the perpendicular drawn to any indefinite straight line from the centre of a given circle be less than the radius of the circle, this straight line shall meet the circle in two points, and two points only, and these points shall be equidistant from the foot of the aforesaid perpendicular.

Let C be the centre of the circle GEF, and let AB be an indefinite straight line such that CD the perpendicular

Fig. 3.



drawn from C to AB is less than the radius of GEF, then AB shall meet GEF in two points, and two points only, equidistant from D.

Because DB is an indefinite straight line, therefore it is always possible to find a point H in DB such that DH is equal to the radius of the circle GEF.

Join CH.

Because CDH is a right angle, therefore CH is greater than DH (Bk. I. Prop. 15),

therefore CH is greater than the radius of the circle GEF,
therefore the point H is situated without the circle GEF
(Bk. II. Prop. 1).

Again, because CD is less than the radius of the circle GEF, therefore the point D is situated within the circle GEF.

Because the circle GEF is a finite closed figure, therefore the straight line DH which joins the point D within this

circle and the point H without it, must meet the circle GEF in some point F between D and H.

Join CF. Then CF is equal to the radius of the circle GEF.

Because a straight line drawn from C to any point in AB between D and F is less than CF, and a straight line drawn from C to any point in AB to the right of F is greater than CF (Bk. I. Prop. 15),

therefore the straight line AB cannot meet the circle in any point to the right of D except the point F.

Similarly it may be proved that the straight line AB meets the circle in one point, as G to the left of D, and in no other point to the left of D except the point G.

Join CG.

Because the two right-angled triangles CGD and CFD have the hypothenuses CG and CF equal to one another, and the side CD common to both,

therefore the remaining sides GD and DF are equal (Bk. I. Prop. 16),

therefore the straight line AB meets the circle in two points, and two points only, equidistant from the point D.

Corollary 1.—It follows from the reasoning of this proposition that, if the length of the perpendicular drawn to any indefinite straight line from the centre of a given circle be equal to the radius of the circle, the straight line will meet the circle in the foot of the perpendicular so drawn, and *in no other point;* and also that if the length of this perpendicular be greater than the radius of the circle, the straight line cannot meet the circle in any point whatever.

Corollary 2.—A straight line cannot cut a circle in more points than two.

Corollary 3.—Every chord of a circle is bisected in the foot of the perpendicular drawn to this chord from the centre of the circle, and the line drawn through the middle point of any chord of a circle perpendicular to this chord passes through the centre of the circle.

DEFINITIONS.

37.—If two circular arcs can be applied to one another in such a manner that every point of each of them coincides with some point of the other, they are said to be *equal* to one another.

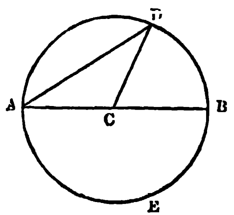
38.—A straight line drawn through the centre of a circle, and having each extremity situated upon the circle, is called a *diameter* of the circle.

PROPOSITION 4.

Every diameter of a circle divides the circle into two equal parts and is greater than any chord not passing through the centre of the circle.

Let ADE be a circle of which C is the centre and AB any diameter.

Fig. 4.



Take any point D in the circle and join AD.

Then AB shall divide the circle into two equal parts, and shall be greater than any chord as AD not passing through the centre.

Let the plane containing the arc ADB be turned round AB until it coincides with the portion of the same plane below AB.

Then the arcs ADB and AEB shall coincide throughout.

For if not, there would be two non-coincident circular arcs having the same centre C and one point as A or B common to both, which is impossible.

Therefore the arcs ADB and AEB coincide throughout, and therefore they are equal to one another (Definition 37).

Again, if CD be joined, AC + CD will be greater than AD.

But CD is equal to CB,

therefore AC + CB is greater than AD,

that is, AB is greater than AD.

DEFINITION.

39.—Every chord not passing through the centre divides the circumference into two arcs, one less than a semicircle, and called the *minor* arc, the other greater than a semicircle, and called the *major* arc; these arcs are also called segments, and the angle between the two straight lines drawn from any point of a segment to its extremities is called *an angle in that segment.*

EXAMPLES.

1. Two equal straight lines are drawn from a given point to the circumference of a circle. Prove that the straight line bisecting the angle between them, produced if necessary, passes through the centre of the circle whether the point be without, within, or upon the circle.

2. Prove that the middle points of a set of parallel straight lines, in a circle, all lie on one straight line passing through the centre, and perpendicular to each of the set of parallel straight lines.

3. Prove that straight lines joining the extremities of parallel chords in a segment are equal to each other: 1st when they join them towards the same parts; 2nd when they join them towards opposite parts.

4. A straight line in the interior of two concentric circles is produced both ways to cut the exterior circle; prove that the parts of this line, intercepted between the two circles, are equal to each other.

5. Prove that the shortest and the longest distances from a given point to a circle lie on the line joining that point with the centre of the circle, or on that line produced, whether the point be within or without the circle.

6. If a point be taken within a circle which is not the centre and straight lines be drawn from this point to the

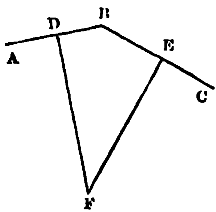
circumference, prove that of these lines one nearer to the shortest line drawn from the given point to the circumference is smaller than one more remote.

PROPOSITION 5.

A circle can always be described so as to pass through three points, not in the same straight line, and only one circle can be so described.

Let A, B, and C, be three points not in the same straight line, then one circle, and only one, can always be described so as to pass through the points A, B, and C.

Fig. 5.



Join AB and BC, bisect these lines in D and E, and through D and E draw DF and EF, at right angles to AB and BC; then these lines must intersect, by Prop. 22, Bk. I.

Because DF bisects AB at right angles, therefore every point in DF is equidistant from A and B.

Similarly, every point in EF is equidistant from B and C; therefore F, the point of intersection of DF and EF, is equidistant from each of the points A, B, and C, and therefore these points are situated on a circle of which F is the centre, or a circle may be described passing through A, B, and C.

Again, only one circle can be so described.

For if any circle pass through A and B its centre will be upon FD, which bisects AB at right angles (Bk. II. Prop. 3, *Cor.* 3), and therefore the centre of every circle passing through A and B is situated upon FD.

Similarly, the centre of every circle passing through B and C is situated upon EF.

Therefore if any two circles pass through A, B, and C,

they must have the same centre, F, and they have one point, as A or B or C, common to both,

therefore they must coincide throughout (Bk. II. Prop. 2).

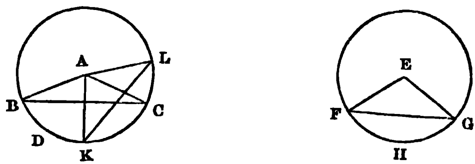
Corollary.—One circle cannot cut another in more points than two, and the straight line joining these two points is bisected at right angles by the straight line joining the centres of the circles.

PROPOSITION 6.

*In equal circles, equal angles at the centres are subtended by equal chords, and equal chords are subtended by equal angles at the centres.**

Let BDC and FHG be equal circles, and BAC and FEG equal angles at the centres, then shall the chords BC and

Fig. 6.



FG be equal to one another. And, conversely, if the chords BC and FG be equal to one another, the angles BAC and FEG at the centres shall be equal to one another.

1. Let BAC be equal to FEG.

Because in the triangles ABC and EFG the sides AB and AC are respectively equal to the sides EF and EG, and the included angles BAC and FEG are equal,

therefore the third sides BC and FG are equal to one another (Bk. I. Prop. 5).

2. Let BC be equal to FG.

* Circles are called *equal* when they have *equal* radii. A circle is often indicated by the letter at its centre: thus the circle of Fig. 6 would be called the circle A.

Because in the triangles ABC and EFG the three sides AB, AC, and BC, are respectively equal to the three sides EF, EG, and FG,

therefore the angle BAC is equal to the angle FEG (Bk. I. Prop. 7).

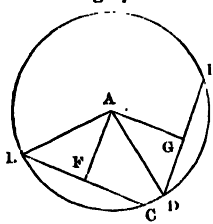
Corollary.—If BAC and KAL be equal angles at A, in the same circle, the proof that BC is equal to KL; and if BC and KL be equal chords in the same circle, the proof that BAC is equal to KAL is in all respects the same as that given above.

PROPOSITION 7.

In the same circle equal chords are equally distant from the centre, and chords which are equally distant from the centre are equal to one another; and similarly for equal circles.

1st. Let BC and DE be two equal chords in the circle BCDE, whose centre is A.

Fig. 7.



Draw AF and AG perpendicular to BC and DE respectively, then BF is equal to one half of BC, and DG is equal to one half of DE (by Bk. II. Prop. 3, *Cor.* 3),

and therefore BF is equal to DG.

Join AB and AD.

Because the two right-angled triangles BFA and DGA have the hypothenuse BA and side BF of the one equal to the hypothenuse DA and side DG of the other, each to each, therefore the remaining side of the one is equal to the remaining side of the other (Bk. I. Prop. 16),

therefore AF is equal to AG.

2nd. Let AF be equal to AG, and repeat the construction. Because the two right-angled triangles ABF and ADG have the hypothenuse AB and side AF of the one equal to the hypothenuse AD and side AG of the other,

therefore BF is equal to DG.

But BC is equal to twice BF, and DE to twice DG,
therefore BC is equal to DE.

The same proof applies in all respects in the case of two different but equal circles.

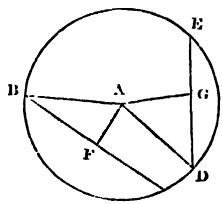
PROPOSITION 8.

In the same circle the greater of two chords is nearer to the centre than the less, and a chord nearer to the centre is greater than one more remote; and similarly for equal circles.

Let BC and DE be two chords in the circle BCDE, of which the centre is A.

1st. Let BC be greater than DE, then shall BC be nearer to the centre than DE.

Fig. 8.



Draw AF and AG perpendicular to BC and DE respectively, then BF is equal to one half of BC, and DG is equal to one half of DE (Bk. II. Prop. 3, *Cor.* 3),

therefore BF is greater than DG.

Join AB and AD.

Because in the two right-angled triangles ABF and ADG the hypothenuses are equal, and the side BF of the one is greater than the side DG of the other, therefore the remaining side of the one is less than the remaining side of the other (Bk. I. Prop. 17),

therefore AF is less than AG.

2nd. Let AF be less than AG, then shall BC be greater than DE.

Make the same construction as before.

Because in the two right-angled triangles, ABF and ADG the hypothenuses are equal, and the side AG of the one is greater than the side AF of the other, therefore the remaining side DG is less than the remaining side BF.

But DE is equal to twice DG, and BC is equal to twice BF,

therefore DE is less than BC.

The same proof applies in all respects in the case of two different but equal circles.

Examples.

1. Prove that the shortest chord which can be drawn to a circle through a given point within it, is perpendicular to the line joining that point with the centre of the circle.

2. Two equal chords are drawn in a circle and produced to the same distance beyond the circle; if the extremities of the lines thus produced be joined, prove that the line which bisects the joining line at right angles passes through the centre of the circle.

3. Through a given point within a circle two equal chords are drawn; prove that the greater and lesser segments, into which these chords are divided, at the given point are equal to each other respectively.

4. A chord, DC, is drawn in a circle and produced to A, so that CA is equal to the radius of the circle. If the diameter AEB be drawn, prove that the angle DEB is three times the angle DAB, E being the centre.

Section II.—On Angles in Circles.

PROPOSITION 9.

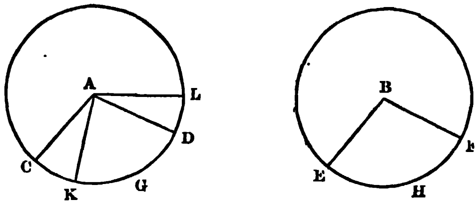
In equal circles equal angles at the centres are subtended by equal arcs, and conversely equal arcs subtend equal angles at the centres.

Let A and B be equal circles, and

1st. Let the angles CAD and EBF at the centres be equal to one another, then shall the arcs CGD and EHF be likewise equal to one another.

Apply the circle A to the circle B so that the point A coincides with the point B, and the straight line AC with the straight line BE.

Fig. 9.



Because A coincides with B, and AC coincides with and is equal to BE, therefore the point C coincides with the point E.

Because the centre A coincides with the centre B, and the point C with the point E, therefore the circles coincide in every point (Bk. II. Prop. 2).

Because AC coincides with BE, and the angle CAD is equal to the angle EBF, therefore AD coincides with BF.

Because A coincides with B, and AD coincides with and is equal to BF, therefore the point D coincides with the point F.

Because the point C coincides with the point E, and the point D with the point F, and the circles coincide in every point, therefore the arc CGD is equal to the arc EHF (Def. 37).

2nd. Let the arc CGD be equal to the arc EHF, then shall the angle CAD be equal to the angle EBF.

Apply the circle A to the circle B so that the point A coincides with the point B, and the line AC with the line BE, then it may be proved as before that C coincides with E, and that the circles coincide in every point.

Because C coincides with E, and the arc CGD coincides with and is equal to the arc EHF, therefore the point D coincides with the point F.

Because A coincides with B, and D with F, therefore the straight line AD coincides with the straight line BF.

Because AC coincides with BE, and AD with BF, therefore the angle CAD coincides with and is equal to the angle EBF.

Corollary 1.—If the angles CAD and KAL at the centre of the *same* circle A be equal to each other, each of the arcs CGD and KDL may be proved to be equal to the arc EHF, and therefore to be equal to one another.

Similarly, if the arcs CGD and KDL of the *same* circle A be equal to each other, each of the angles CAD and KAL may be proved to be equal to the angle EBF, and therefore to be equal to each other.

Corollary 2.—It follows from this Proposition and Proposition 6, that in equal circles equal arcs are subtended by equal chords, and equal chords by equal arcs.

PROPOSITION 10.

If an angle at the centre of a circle, and an angle at the circumference, be subtended by the same arc, the angle at the centre shall be double of that at the circumference.

Let ABC be a circle and O its centre, and let BOC be an angle at the centre O, and BAC an angle at the point A of

Fig. 10.

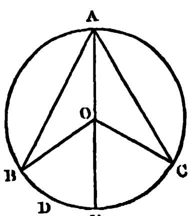
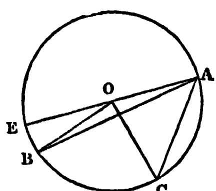


Fig. 11.



the circumference, subtended by the same arc BDC, then shall the angle BOC be double of the angle BAC.

Join AO, and produce it to meet the circumference in E.

First, let B and C lie on opposite sides of AE, as in Fig. 10.

Because AO is equal to BO, the angle OAB is equal to the angle OBA, and therefore OAB+OBA is double of OAB.

Because the side AO of the triangle AOB is produced to E, the exterior angle EOB is equal to OAB+OBA,

therefore EOB is double of OAB.

Similarly, EOC is double of OAC,

therefore EOB+EOC is double of OAB+OAC,

that is, BOC is double of BAC.

Next, let B and C lie on the same side of AE, as in Fig. 11.

It may be proved as before that

EOB is double of OAB,

and that EOC is double of OAC,

therefore EOC ~ EOB is double of OAC ~ OAB,

that is, BOC is double of BAC.

Corollary.—If one of the equal sides of an isosceles triangle be produced through the vertex, the exterior angle is double of either of the interior non-adjacent angles.

PROPOSITION II.

The angles in the same segment of a circle are equal to one another.

Let BAC and BDC be angles in the same segment of the

Fig. 12.

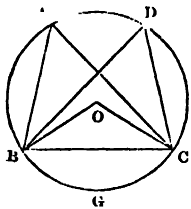
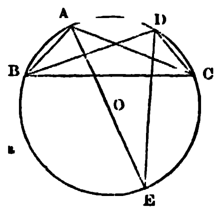


Fig. 13.



circle ABC, cut off by the chord BC, then shall the angle BAC be equal to the angle BDC.

First let the segment BADC be greater than a semicircle, as in Fig. 12.

Because the two angles, viz. BOC at the centre and BAC at the circumference of the circle O, are subtended by the same arc BGC, therefore the angle BOC is double of the angle BAC.

For the same reason the angle BOC is double of the angle BDC.

therefore the angle BAC is equal to the angle BDC.

Next, let BADC be less than a semicircle, as in Fig. 13.

Join AO, and produce it to meet the circumference in E.

Because AE is a diameter, therefore BACE is greater than a semicircle, and therefore the angle BAE is equal to the angle BDE by the first part of this proposition.

Similarly, the angle EAC is equal to the angle EDC,

therefore BAE+EAC is equal to BDE+EDC,

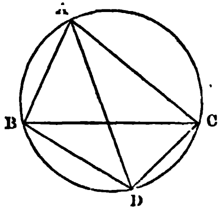
that is, the angle BAC is equal to the angle BDC.

PROPOSITION 12.

The angles in the major and minor segments of a circle, cut off by the same chord, are together equal to two right angles.

Let BAC and BDC be two angles in the major and minor segments respectively of the circle ABC, cut off by the same chord BC, then shall BAC, together with BDC, be equal to two right angles.

Fig. 14.



Join AD.

Because BDA and BCA are angles in the same segment of the circle ACDB, therefore they are equal to one another.

Similarly, the angles ADC and ABC are equal to one another.

therefore BDA+ADC is equal to ACB+ABC,

that is, BDC is equal to ACB+ABC,

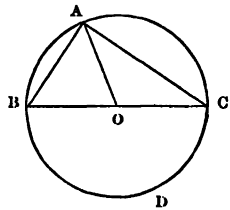
therefore BDC+BAC is equal to ACB+ABC+BAC,
therefore BDC+BAC is equal to two right angles
(Bk. I. Prop. 23).

PROPOSITION 13.

The angle in a semicircle is a right angle, and the angle in a segment greater than a semicircle is less than a right angle, and the angle in a segment less than a semicircle is greater than a right angle.

Let ABDC be a circle and BC its diameter, and therefore BAC a semicircle, then shall the angle BAC be a right angle.

Fig. 15.



Let O be the centre of the circle and join OA.

Because OA is equal to OB, therefore the angle OAB is equal to half the angle AOC (Bk. II. Prop. 10, *Cor.*).

Similarly, the angle OAC is equal to half the angle AOB,

therefore OAB+OAC is half of AOC+AOB.

But OAB+OAC is equal to BAC,
and AOC+AOB is equal to two right angles (Bk. I. Prop. 11), therefore BAC is equal to one right angle.

Again, because BAC+ABC is less than two right angles (Bk. I. Prop. 14), and BAC is one right angle,

therefore ABC is less than one right angle,

therefore the angle ABC in the segment ABDC greater than a semicircle is less than a right angle.

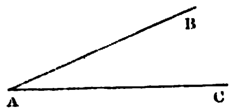
Because the angles in any major and minor segments cut off by the same chord are together equal to two right angles.

And because the angle in any major segment has just been proved to be less than one right angle.

Therefore the angle in any minor segment, that is, in any segment less than a semicircle, is greater than one right angle.

Note.—By Definition 10 the angle BAC is the smallest amount of turning about A which is required to bring either of the straight lines AB or AC into coincidence with the other line, the revolving line always remaining in the same plane.

Fig. 16.



If the restriction imposed by the word *smallest* be removed we have a more general definition of an angle which is sometimes useful, and with this extended definition it is easily seen that there are an infinite number of angles between two given straight lines, as AB and AC, and that it is necessary in each case to specify the particular mode of description contemplated.

Thus, to recur to the illustration given in the notes to the Introduction, suppose that AB and AC are the two hands of a watch, and that AB points to XII and AC to II; then AB may be moved into the position AC passing over the ten minute divisions between XII and II, and the angle thus described is the angle meant when the angle BAC is spoken of without further description. But AB may be also brought into the position AC by revolving in the opposite direction through the hours XI, X, &c., thus passing over fifty minute divisions in its revolution from AB to AC. Or again, AB may be brought into the position AC by revolving as in the first case from XII to II, and then continuing to revolve in the same direction until it has gone completely round the circle and returned to II again, thus passing over seventy minute divisions in its course; and so on.

Whenever the angle between two straight lines is mentioned without any further specification of the way in which this angle is described, it is always understood, as we have said, that the shortest and most direct method of description is contemplated. Such are the angles considered in all the propositions hitherto proved, and such will be the angles which may occur in any future proposition unless the contrary be expressly mentioned.

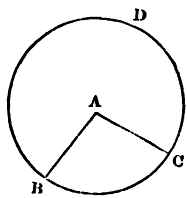
There are, however, certain propositions concerning angles which are true whatever be the method of description, and we proceed, by way of example, to the consideration of one of these propositions, premising the following definition.

DEFINITION.

40.—If a point be taken within or upon the circumference of a circle, and a straight line be made to revolve round this point through any angle meeting in its revolution different points in succession in an arc of the circle, the angle so described is said to be *the angle at that point subtended by that arc of the circle.*

Thus, if A be the given point within the circle BCD, and a straight line revolve round A from AB to AC meeting successive points of the arc BC, the angle so described is called *the angle at A or the angle BAC subtended by the arc BC.*

Fig. 17.



If the revolving line met successive points of the arc BDC the angle would be called *the angle at A or the angle BAC subtended by the arc BDC.*

It is clear that in accordance with what has been already stated the angle in the first case might be equally well described as the angle BAC *simply*, but that in the second case the addition of the words 'subtended by the arc BDC' is absolutely necessary.

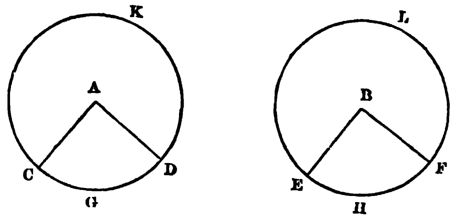
PROPOSITION 9 A.

In equal circles equal angles at the centres are subtended by equal arcs, and equal arcs are subtended by equal angles at the centres.

Referring to Proposition 9 we have already proved that

if the angles at A and B subtended by the arcs CGD and EHF respectively be equal to each other, then the arcs CGD and EHF will be also equal to one another.

Fig. 18.



Now the angles at A and B subtended by the whole circumferences are equal to one another, each of them being equal to four right angles, by Bk. I. Prop. 11, *Cor.* 3.

Also, the whole circumferences of the circles A and B are equal to one another.

Therefore the angles at A and B subtended by the arcs CKD and ELF are equal to one another, and the subtending arcs CKD and ELF are also equal to one another.

Therefore the proposition is true for angles greater than two and less than four right angles.

Also, whenever the angles at the centres of the circles are each increased by four right angles the subtending arcs are each increased by a whole circumference.

Therefore the proposition is true for angles of any magnitude whatever.

Examples.

1. Through a point C on a circle two straight lines ACB, DCE are drawn, cutting the circle in B and E: prove that the straight line which bisects the angles ACE, DCB meets the circle in a point F equidistant from B and E.

2. Two straight lines AB, CD cut one another in the

point E within a given circle: prove that the sum of the angles, subtended at the centre by the arcs AC and BD, is double of the angle AEC.

3. If the point E be without the circle, prove that the difference of the angles, subtended at the centre by the arcs AC and BD, is double of the angle AEC.

4. A circle is drawn circumscribing a given equilateral triangle: prove that the sum of the distances of any point on this circle from the two nearer angles is equal to the distance of that point from the remaining angle.

5. Of all the triangles on the same base, and between the same parallels, the isosceles triangle is that which has the greatest vertical angle.

6. From the angles A, B, and C of a triangle ABC perpendiculars AD, BE, and CF are drawn to the opposite sides, and the triangle DEF is formed: prove that the angles of this triangle at D, E, and F are bisected by AD, BE and CF respectively.

7. ABCD is a quadrilateral figure inscribed in a circle, and O is the point of intersection of its diagonals; through O a chord EOF is drawn, such that O is its middle point. Prove that O is also the middle point of the part of this chord intercepted between any two opposite sides of the quadrilateral.

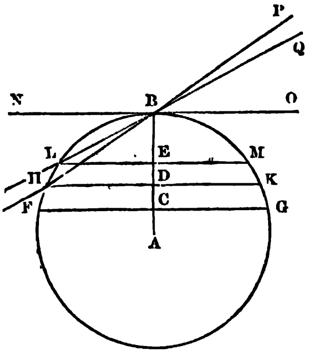
SECTION III.—ON THE CONTACT OF STRAIGHT LINES AND CIRCLES WITH CIRCLES.

Note:—Let FBG be a circle, A its centre, and AB any radius.

In AB take any number of points as C, D, E, &c., and through these points draw straight lines perpendicular to AB and each therefore meeting the circle in two points as F and G, H and K, L and M, &c., equidistant from C, D, E, &c. (Bk. II. Prop. 3).

Then we know that FG is greater than HK, and HK greater than LM, and so on (Bk. II. Prop. 8); also that if the line NBO be drawn through B perpendicular to AB it will meet the circle in no point whatever besides B (Bk. II. Prop. 3, *Cor.* 1).

Fig. 19.



The straight line NBO which meets the circle A in the point B, and in that point only, is frequently defined as being the tangent to the circle A at the point B.

A more instructive definition of the tangent, and which leads, as we shall show (Prop. 14), to the same result, may be obtained from the following considerations.

Suppose a straight line to revolve round the point B so as to coincide in succession with the lines HBP, LBQ, and so on, and therefore so as to meet the circle A in the fixed point B, and in a second point which is variable and which moves along the arc FLB from H towards B, then the ultimate position assumed by the moving line as the variable point moves up to, and ultimately coincides with the fixed point B, is defined as being the tangent to the circle A at the point B.

Hence we adopt the following general definition of the tangent to a circle at any point.

DEFINITION.

41.—If a straight line cut a circle in two points and be made to revolve round one of these points until the other point approaches indefinitely near to and ultimately coincides

with the first point, the line with which the moving line ultimately coincides is called the *tangent* to the circle at that first point.

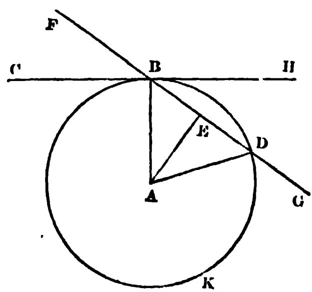
When a straight line meets a circle in two points it is called a *secant* to the circle at each of these points.

PROPOSITION 14.

The tangent to a circle at any point is perpendicular to the radius drawn from the centre to that point, and cannot meet the circle in any other point.

Let BDK be a circle and A the centre, and FBDG a straight line cutting the circle in B and some other point, as D, and let CBH be the ultimate position of FG, as D moves up to B.

Fig. 20.



Bisect BD in E and join AB, AD, and AE, then AE is at right angles to FBDG (Bk. II. Prop. 3, *Cor.* 3).

Because AE is always perpendicular to FG therefore it is perpendicular to FG in its ultimate position.

Because D coincides with B ultimately, therefore AD and AB ultimately coincide with each other, and therefore AE, which lies between them, must coincide with either of them, as AB.

Therefore AB is perpendicular to FG in its ultimate position.

Therefore AB is perpendicular to CH.

Also, because the perpendicular distance AB of the straight line CH from the centre of the circle is equal to the radius of the circle, therefore the straight line CH cannot meet the circle in any other point besides B (Bk. II. Prop. 3. *Cor.* 1).

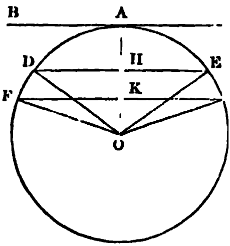
Corollary.—If a line be drawn through any point of a circle perpendicular to the tangent at that point, this line will pass through the centre of the circle.

PROPOSITION 15.

The tangent at any point of a circle is parallel to the chords which are bisected by the radius drawn to the point of contact, and any pair of parallel chords in a circle cut off equal arcs.

Let BC be a tangent, at the point A, to the circle DAE, whose centre is O.

Fig. 21.



Let DE be a chord bisected by AO in H, then BC shall be parallel to DE.

Because OHA bisects DE, the angles at H are right angles (Bk. II. Prop. 3, *Cor.* 3).

Because BAC is the tangent at A, the angles at A are right angles (Bk. II. Prop. 14).

Therefore BC and DE are each of them at right angles to OA.

Therefore BC is parallel to DE.

Let FG be any other chord bisected by OA, then the arc FD shall be equal to the arc EG.

Because the angles DOH and EOH are equal to each other, as also the angles FOH and GOH, therefore the difference of FOH and DOH is equal to the difference of GOH and EOH,

therefore FOD is equal to EOG,

therefore the arc FD is equal to the arc EG (Bk. II. Prop. 9).

PROPOSITION 16.

If two circles have one point common to both, and not situated in the straight line joining the centres of the circles, or that line

produced, then they shall have a second point in common, and the distance between their centres shall be less than the sum and greater than the difference of their radii.

Let the circles, whose centres are A and B, have a point C common to both, and such that C is not situated in the straight line AB, or that line produced.

Fig. 22,

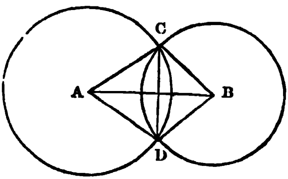
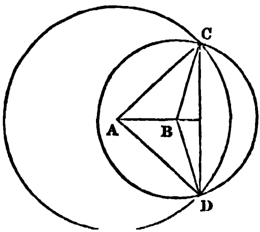


Fig. 23.



Join AC, BC, and AB, and let BD be a straight line through B, making with AB the angle ABD, equal to the angle ABC on the other side of AB. Let BD meet the circle B in D and join AD.

Because in the two triangles ABC and ABD the two sides AB and BC are respectively equal to the two sides AB and BD, and the included angles are equal, therefore AC is equal to AD, or the point D is a point on the circle A.

But D is also a point on the circle B,

therefore D is common to both circles.

And because ABC is a triangle,
therefore AB is less than AC+CB, and greater than AC ~ CB, i.e. the distance between the centres is less than the sum and greater than the difference of the radii.

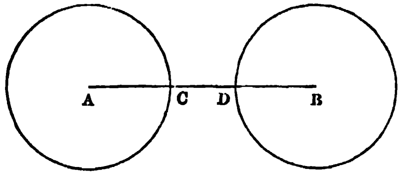
PROPOSITION 17.

If two circles be so situated that every point of each of them is without the other, then the distance between the centres of the circles shall be greater than the sum of their radii, and if the

circles be so situated that every point of one of them is within the other, then the distance between their centres shall be less than the difference of their radii.

1st. Let A and B be two circles so situated that every point of each of them is without the other, and let AB be

Fig. 24.



joined, then shall AB be greater than the sum of the radii of the two circles.

Let the straight line AB cut the circles A and B in the points C and D respectively.

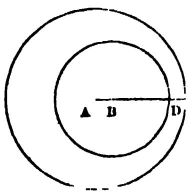
Because every point of B is without A, therefore AD is greater than AC.

To each of these unequal lines add DB,

therefore AD + DB is greater than AC + DB;

that is, AB is greater than the sum of the radii of A and B.

Fig. 25.



2nd. Let A and B be two circles so situated that every point of one of them, as B, is within the other A, and let AB be joined as before, then shall the distance AB be less than the difference of the radii of A and B.

Produce AB to meet the circles A and B in the points C and D respectively.

Because every point of B is within A, therefore AD is less than AC.

From each of the unequal lines AD and AC take the line BD,

therefore AD ~ BD is less than AC ~ BD,

that is, AB is less than the difference of the radii of A and B.

Note.—If the circles of Proposition 16, or one of them, be moved so that the distance AB is gradually *increased* in Fig. 22, or gradually *diminished* in Fig. 23, it is clear that the distance CD between the points of section will continually diminish.

When these points C and D coincide in one point, as at

Fig. 26.

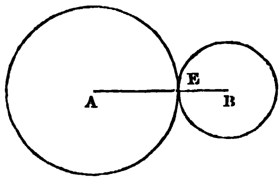
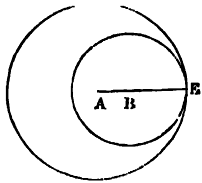


Fig. 27.



E in Figs. 26 and 27, the circles are said to touch each other in the point E. Hence we have this general definition.

DEFINITION.

42.—If two circles cut one another and be moved in such a way that the two points of intersection continually approach each other and ultimately coincide, the circles are said to touch each other at the point in which these two points ultimately coincide.

PROPOSITION 18.

If two circles touch one another the point of contact shall be situated upon the straight line joining their centres, or upon that line produced, and the distance between their centres shall be equal to the sum or difference of the radii.

Let the two circles of Proposition 16, whose centres are A and B, be moved in such a manner that the distance CD between their points of intersection C and D continually diminishes and ultimately vanishes, then by Definition 42 the two circles will touch each other in that point with which C and D ultimately coincide, as in Fig. 26 or Fig. 27 above.

But CD is always bisected in the point in which it is met by AB or AB produced (Bk. II. Prop. 5. *Cor.*), and therefore the points C and D must ultimately coincide with some point E in the line AB, as in Fig. 26 or Fig. 27.

Therefore the point of contact of two circles which touch each other lies in the straight line joining their centres, or in that line produced,

therefore if the point B be *without* the circle A, as in Fig. 26,

AB is equal to AE + EB,

and if the point B be *within* the circle A, as in Fig. 27,

AB is equal to AE ~ EB.

Or the distance between the centres is equal either to the sum or difference of the radii of the circles.

PROPOSITION 19.

If two circles touch one another in any point they cannot have any other common point.

Let the circles A and B touch one another in the point E, they cannot have any other common point besides E.

Fig. 28.

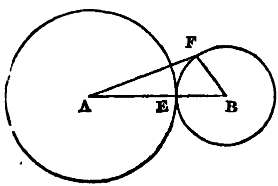
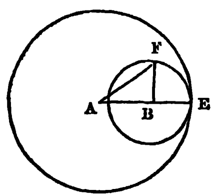


Fig. 29.



First, let the centre B be without the circle A, as in Fig. 28.

Because the circles touch one another in E, therefore E is on the straight line AB (Bk. II. Prop. 18).

Let F be any point different from E situated on the circle B and join AF and FB.

Because AF+FB is greater than AB, take away from these unequals the equal lines FB and EB respectively,

therefore AF is greater than AB ~ EB,

that is, AF is greater than AE.

But AE is the radius of the circle A,

therefore F is not situated upon the circle A.

Next, let the centre B be within the circle A, as in Fig. 29.

Then, E must be on the straight line AB produced. Let F be a point on the circle B and join AF and FB.

Because AF is less than AB+BF,

and BF is equal to BE,

therefore AF is less than AB+BE,

that is, AF is less than AE.

But AE is the radius of the circle A,

therefore AF is less than the radius of A,

therefore F is not a point on the circle A.

Corollaries to Propositions 16, 17, 18, *and* 19.

Corollary 1.—If the distance between the centres of two circles be *less than the sum and greater than the difference of the radii* of the circles, the circles must cut each other in two points, for they cannot lie without each other or one within the other (by Prop. 17), and they cannot touch each other in one point only (by Prop. 18).

Corollary 2.—If the distance between the centres be *greater than the sum or less than the difference of the radii*, they must lie each wholly without the other in the first case, and one within the other in the second case, for they cannot cut one another in two points (by Prop. 16), and they cannot touch each other in one point only (by Prop. 18).

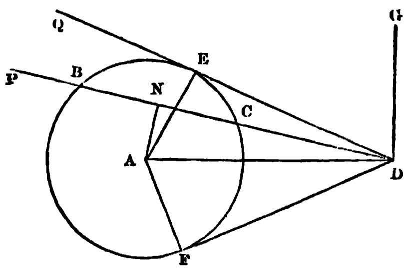
Corollary 3.—If the distance between the centres be *equal to the sum or difference of the radii* they must touch one another in one point only, for they cannot cut one another (by Prop. 16), and they cannot lie each without the other, or one within the other (by Prop. 17).

PROPOSITION 20.

Two tangents may be always drawn to a given circle, from a given point without it, and these tangents will be equal to each other, and equally inclined to the line joining the centre of the circle with the given point.

Let BCF be a circle, A its centre, and D a point without it, then it is always possible to draw from D two tangents to the circle BCF, and these tangents will be equal to each other in length, and equally inclined to the straight line AD.

Fig. 30.



Let the indefinite straight line DP revolve round D from the position of coincidence with DA to the position of coincidence with DG, at right angles to DA, and let AN be the perpendicular dropped from A upon DP in any position.

When DP coincides with DA the length of AN is clearly zero, and when DP coincides with DG the length of AN is equal to AD.

Therefore as DP revolves through a right angle the length of the perpendicular upon it from A increases from zero to AD as a limit.

But AD is greater than the radius of the given circle.

Therefore in some intermediate position the length of the perpendicular from A upon DP must be equal to the radius of the circle.

Let DQ be the position of DP in this case.

Draw AE perpendicular to DQ.

Because AE is equal to the radius of the circle, therefore E is a point on the circumference.

But QED is drawn through E at right angles to AE,

therefore QED touches the circle,

and it passes through D by hypothesis.

By similar reasoning another line DF may be drawn through D touching the circle in the point F on the other side of AD.

Then in the two right-angled triangles DEA and DFA we have

AE equal to AF and the hypothenuse AD common,

therefore the triangles are equal in all respects,

therefore DE is equal to DF,

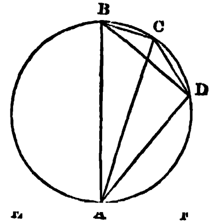
and ADE is equal to ADF.

PROPOSITION 21.

If a straight line touch a circle and from the point of contact another straight line be drawn cutting the circle, the angles which the cutting line makes with the opposite parts of the touching line shall be equal to the angles in the alternate segments of the circle.

Through the point A of the circle ABC let the straight line EF be drawn touching the circle, and from A let AC be drawn cutting the circle, then shall the angles EAC and FAC be equal to the angles in the segments CDA and CBA respectively.

Fig. 31.



From A draw AB at right angles to EF meeting the circle again in B, join BC, take any point D in the segment CDA, and join DA, DC, and DB.

Because EF touches the circle at the point A, and AB is at right angles to EF,

therefore AB coincides with the diameter of the circle passing through the point A (Bk. II. Prop. 14, *Cor.*),

therefore BCA is a semicircle.

Because BCA is a semicircle therefore the angle ACB is a right angle.

Because ACB is a right angle therefore CBA+BAC is equal to a right angle,

therefore CBA+BAC is equal to BAF,

therefore CBA is equal to BAF~BAC,

that is, CBA is equal to CAF,

or CAF is equal to any angle in the segment CBA.

Because CBA+CDA is equal to two right angles (Bk. II. Prop. 12),

and also CAF+CAE is equal to two right angles (Bk. I. Prop. 11),

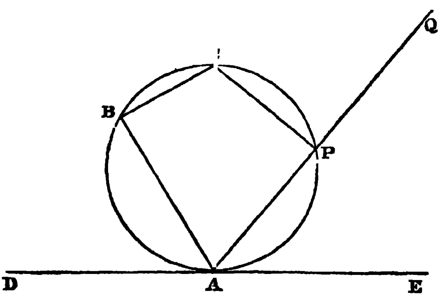
therefore CBA+CDA is equal to CAF+CAE,

but CBA is equal to CAF,

therefore CDA is equal to CAE.

Note.—Another very suggestive and instructive proof of the foregoing proposition may be derived from a direct application of the definition of the tangent.

Fig. 31 *a.*



Let ABC be a circle, and B and C given points on the circumference, and DAE the tangent at A.

Join AB and BC, and through A draw a straight line APQ cutting the circle in the points P and A.

Then the angle BCP is, by Prop. 12, the supplement of the angle BAQ.

Let the line AB revolve round the point A, as in Def. 41, until it coincides with the tangent AE, then the angles BCP and BAP coincide, in the limit, with the angles BCA and BAE respectively.

Therefore the angle BCA is the supplement of the angle BAE.

That is, the angle BCA is equal to the angle BAD.

EXAMPLES.

1. Two circles cut one another in the points A and B. AC and AD are drawn, each touching one circle, and terminated by the other circle in the points C and D. Prove that the triangles ABC and ABD are equiangular to one another.

2. On the same base and on the same side of it there are two segments of circles, of which ACB is a semicircle and ADB a quadrant. Through P, any point in ADB, the line APQ is drawn, cutting ACB in Q, and QB is joined. Prove that PQ is equal to QB.

3. ACB and ADB are two segments of circles on the same base AB; AC and BC are two lines from A and B meeting at C on the segment ACB, and produced to meet the segment ADB in D and E respectively. Prove that wherever C may be the arc DE is of constant length.

4. If two circles intersect, prove that the common chord, when produced, bisects each of the common tangents.

5. If two circles touch each other externally, and parallel diameters be drawn one to each circle, prove that the straight line which joins the extremities of these diameters towards opposite parts passes through the point of contact.

6. ABC is a triangle inscribed in a circle O whose centre is joined to the middle point D of the arc BC, and AD is drawn. Prove that the angle ADO is half the difference of the angles B and C.

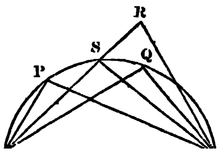
SECTION IV.—ON LOCI CONNECTED WITH THE CIRCLE.

PROPOSITION 22.

The locus of a point at which a given straight line subtends a given angle is two circular arcs, each passing through the extremities of the given line.

Let BC be the given straight line and P a point on the required locus, and therefore such that the angle BPC is equal to the given angle.

Fig. 32.



Through the points B, P, and C, one, and only one, circumference can be drawn (Bk. II. Prop. 5, *Cor.*); let this be BPC.

Because the angles in the same segment of a circle are equal to one another, if any point Q, different from P, be taken in the circumference BPC, the angle BQC will be equal to the angle BPC,

therefore every point on the arc BPC is a point of the required locus.

Again, let any other point be taken, as R, *without* the arc BPC and let S be the point in which the line BR meets this arc.

Then because the exterior angle of a triangle is greater than the interior non-adjacent angle, therefore BSC is greater than BRC.

Therefore BRC is *less* than the given angle.

Therefore no point above BC and *without* the arc BPC can be on the required locus.

In like manner, it may be shown that no point above BC and *within* the arc BPC can be on the required locus.

Therefore no point above BC, and *not* situated on the arc BPC, can be on the required locus.

An arc equal to BPC can be described on BC and below BC.

Therefore the locus in question is two circular arcs, each passing through B and C.

Corollary.—When the given angle is a right angle the locus is the circle of which BC is a diameter.

PROPOSITION 23.

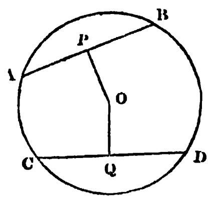
The locus of the middle points of equal chords in a given circle is another concentric circle.

Let O be the centre of the circle.

Let AB be one of the chords of given length, and P its middle point.

Let CD be any other such chord, and Q its middle point.

Fig. 33.



Join OP and OQ, then these lines are perpendicular to AB and CD respectively (Bk. II. Prop. 3, *Cor.* 3).

But equal chords in a circle are at equal distances from the centre,

therefore OP is equal to OQ.

Therefore the middle point of every chord equal to AB is on the circumference of a circle passing through P, and having its centre at O, and conversely because lines equally distant from the centre are equal, it follows that all points on this circle are middle points of equal chords.

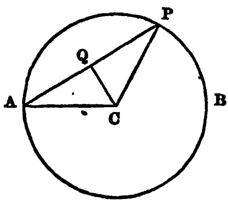
PROPOSITION 24.

The locus of the middle points of all chords drawn through a given point in the circumference of a given circle is a circle passing through the given point and having the radius of the given circle to the given point for a diameter.

Let APB be the given circle, C the centre, and A the given point on the circumference.

Let AP be any chord through A. Bisect AP in Q and join CQ.

Fig. 34.



AC is equal to CP, therefore ACP is an isosceles triangle, and therefore CQ, the line joining the vertex with the bisection of the base, is perpendicular to AP.

Therefore the locus of Q is the locus of a point at which a given straight line AC subtends a right angle.

Therefore the locus of Q is the circle of which AC is the diameter (Bk. II. Prop. 22, *Cor.*).

EXAMPLES.

1. A straight line of constant length is always parallel to a given line, and one of its extremities describes a given circumference. Prove that the locus of the other extremity is a circle whose centre is in the given line.

2. Prove that the locus of the middle points of all the straight lines drawn from a given point to a given circle is another circle.

3. Prove that the locus of the middle points of all chords to a circle drawn through a given point is another circle.

4. A ladder is raised gradually against a wall; prove that the locus of the middle point is a circle; find its centre and radius.

5. A circle rolls within another fixed circle whose radius is equal to its own diameter. Prove that the locus of a point on the circumference of the rolling circle is a diameter of the fixed circle.

6. Prove that the locus of the middle point of a straight line, which moves in such a way that the sum of the perpendiculars upon it from two fixed points is constant, is a circle, and find the centre of this circle.

7. Through one extremity of the diameter of a semicircle lines are drawn and produced beyond the semicircle, so that the part produced is equal to the chord joining the point of intersection of the line and semicircle with the other extremity of the diameter. Find the locus of the extremities of the parts so produced.

Note.—When we speak of one straight line as being *equal* to another, or, more accurately, as being *equal in length* to another, we have a very exact and precise idea of what is meant, viz. that the one line can be applied to the other so that the extremities and every intermediate point of the one may coincide with the extremities and some intermediate point of the other. So, again, when we speak of one circular arc as being *equal* to another circular arc of *equal radius*, we mean that the one arc can be applied to the other, so that the extremities and every intermediate point of the one may coincide with the extremities and some intermediate point of the other. But our ideas of equality of length are not confined to lines which may be superposed upon one another in this manner, and in practice we speak familiarly of the *lengths* of curves or of circular arcs, meaning thereby the lengths of straight lines equal in length to these curves or circular arcs, although a straight line and the arc of a circle cannot of course be superposed one upon the other. It is well therefore to have an accurate definition of the length of a curved line, which we will now proceed to investigate.

If a circle be made to roll along a straight line, as the wheel of a coach rolls along a road, every point of the circle in each revolution will coincide in succession with every point of the straight line lying between the first and last points of contact, and the length of the straight line between these points is said to be the length of the circle, or of the circumference of the circle. In the same way, if the length of the line be measured before the circle has made a complete revolution—that is, before the point of the circle first of all in contact with the line is brought into coincidence

with a point of the line again—the length of this smaller line is called the length of the arc of the circle between the points of such first and last contact with the line.

In this motion it is supposed that no point of the circle *slides* along any portion of the line, but that successive points of the circle are brought into contact successively with fresh points of the line, and that fresh points of the line are brought into contact successively with fresh points of the circle.

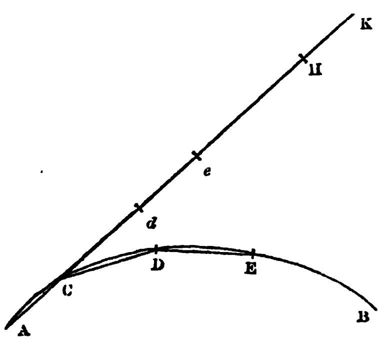
It is clear that the straight line may be supposed to roll on the circle as well as the circle on the straight line.

Hence we have this general definition, not only applicable to a circle or circular arc, but to any curved line, viz.:

DEFINITION.

43.—If a straight line be made to roll upon a curved line, the length of the straight line between its first and last points of contact with the curved line, is defined to be the length of the curved line between its first and last points of contact with the straight line.

Fig. 35.



Again, suppose AB to be a curved line, and C, D, E, &c., successive points upon it, intermediate between A and B, and let the chords AC, CD, DE, &c., be drawn.

Let AH be any indefinite straight line, and let one of its extremities be made to coincide with the point A, the line itself coinciding with AC produced.

Let the straight line AH revolve about the point C until it coincides with CD, and let *d* be the point of AH which coincides with the point D in this position.

Again, let AH revolve about D or *d* until it coincides with DE, and let *e* be the point upon it which coincides with E in this position ; and so on.

It is clear that if H be the point with which B coincides in the final position of the line, AH will be equal to the sum of the lengths of the chords AC, CD, DE, &c.

Now let the number of intermediate points C, D, E, &c., be indefinitely increased, and let the distance between each successive pair of points be indefinitely diminished, then since this may be done without limit it follows that *ultimately* there will be no point on the curved line AB which is not brought into coincidence with some point on the movable straight line in the manner described on page 93, and therefore if K be that point upon the straight line which ultimately coincides with the point B of the curve it follows that the length of AK will be the length of the curve AB.

Therefore the length of the curved line AB will be the same as the sum of the lengths of the chords AC, CD, &c., when the number of these chords is indefinitely increased and the length of each is indefinitely diminished ; whence we have this proposition.

If a series of points be taken between the first and last points on any finite curved line, and the chords between each pair of points in succession be drawn, and if the number of such points be indefinitely increased and the length of each chord be indefinitely diminished, the ultimate sum *of the lengths of these chords is the length of the curved line.*

BOOK III.

PROBLEMS OF CONSTRUCTION CONNECTED WITH THE STRAIGHT LINE AND CIRCLE.

Our investigations hitherto have been purely *theoretical*, and have been confined to the demonstration of certain properties of figures, assumed to exist, satisfying certain proposed conditions. Such investigations are called theorems. We now pass to the more *practical* portion of the subject, that is to say, the determination of methods by means of which such figures are to be drawn, or rather approximately drawn, for, owing to the imperfection of our instruments, the ideal state contemplated in theoretical geometry cannot be attained by them.

Many instruments have been invented by means of which certain constructions can be effected, such as squares, parallel rulers, elliptic compasses, graduated sectors of circles, and so forth; but in elementary geometry we suppose ourselves confined to the use of a *ruler*, by means of which straight lines may be drawn from one point to another, or given lines may be produced if necessary, and *compasses*, by means of which distances may be measured off from given larger lines equal to given smaller lines, and circles may be described round given points as centres, and having lines of given length for their radii.

The determination of the method of constructing a given figure with given instruments is called a *problem*, and the *solution* of a problem requires us to *show how* the required construction can be effected by the use of these instruments, and to *prove that* the construction so effected is correct.

In solving a problem, as in investigating a locus, it will generally be most convenient to assume the thing done and to draw a figure assumed to be the required figure, and then from theoretical geometry to deduce certain properties from this figure, whence the steps in the construction may be made evident. Such a method of solution is called the *analytical* method. When, on the contrary, we do not assume the solution, but enunciate the steps one after the other, adding step to step until we arrive at the required result, the method adopted is called the *synthetical* method. In the earlier problems which follow (viz. from Problem 1 to Problem 11 inclusive), the synthetical method is adopted, the successive steps being almost self-evident, but in the later problems the analytical method has been employed.

It is assumed here that the student is already familiar with the compasses and ruler, and the method of using these instruments, and it is therefore considered sufficient to direct that given points be joined, or given finite lines be produced, or given distances be measured off upon given lines, or circles be described with given centres and radii, without any further description of the way in which these things are effected, and any proposed problem is regarded as solved when it has been shown how the required result may be attained by successive steps of this nature.

PROBLEM 1.

Through a given point in a given straight line to draw a straight line making a given angle with the given straight line.

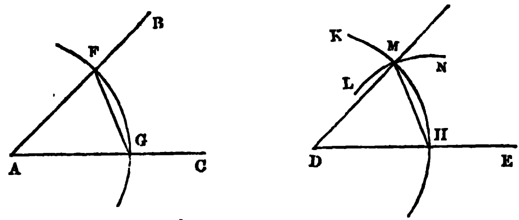
Let BAC be the given angle contained at the point A by the indefinite straight lines AB and AC.

Let DE be the given straight line and D the given point in it, through which it is required to draw a line making with DE an angle equal to BAC.

With centre A and any radius, describe a circle cutting AB and AC in the points F and G respectively.

With centre D and radius equal to AF, describe the circle KMH, cutting DE in H.

Fig. 1.



With centre H and radius equal to the distance between F and G, describe the circle LMN, which will cut the circle KMH in one point (as M) above DE (Bk. II. Props. 16 to 19, *Cor.* 1).

Join DM, then DM or DM produced will be the line required.

Join FG and MH.

Because the circular arcs HM and FG have equal radii, and equal chords,

therefore the angles at the centres CAB and MDE are equal (Bk. II. Prop. 6).

If the circles KMH and LMN be completed there will be another point of intersection below DE, giving a second line fulfilling the conditions.

PROBLEM 2.

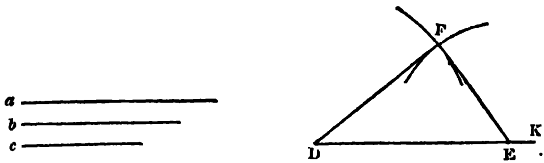
Given three straight lines such that any two of them are together greater than the third. To construct a triangle having its sides respectively equal to these three given lines.

Let *a*, *b*, and *c*, be the three given straight lines, such that any two of them are together greater than the third.

It is required to construct a triangle having its sides respectively equal to *a*, *b*, and *c*.

Take any indefinite straight line DK, and cut off DE equal to *a*.

Fig. 2.



With centre D and radius equal to *b* describe a circle, and with centre E and radius equal to *c* describe another circle, then these circles will cut each other in two points on opposite sides of DE, because DE is less than the sum and greater than the difference of their radii.

Let F be one of these points, and join DF and FE.

Then DFE will be the triangle required.

The proposition requires no proof.

If any one of the straight lines *a*, or *b*, or *c*, were greater than the sum of the other two, the two circles described as above would *not* cut one another and the construction would fail.

Also, we know by Prop. 1, Bk. I., that no triangle can be drawn in such a case satisfying the proposed conditions.

PROBLEM 3.

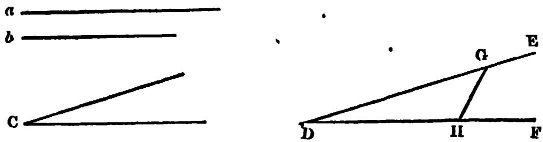
Given two finite straight lines and an angle. To construct a triangle having two of its sides respectively equal to the two given straight lines, and the angle included by these sides equal to the given angle.

Let *a* and *b* be the two given straight lines, and C the given angle.

It is required to construct a triangle having two of its sides equal to *a* and *b* respectively, and the angle included by these sides equal to the angle C.

Draw two indefinite straight lines, DE and DF, inclined to each other at the angle EDF equal to C (Prob. 1).

Fig. 3.



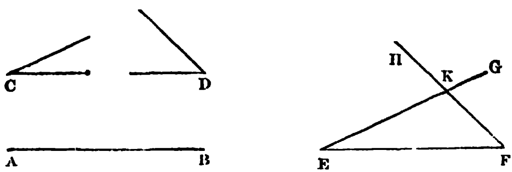
Cut off DG and DH equal to *a* and *b* respectively.
Join GH, then DGH is the triangle required.
The proposition requires no proof.

PROBLEM 4.

Given a straight line and two angles. To construct a triangle having one of its sides equal to the given line, and the angles at the extremities of this side equal respectively to the two given angles.

Let AB be the given straight line, and C and D the given angles. It is required to construct a triangle having one of

Fig. 4.



its sides equal to AB, and the angles at the extremities of this side equal to C and D respectively.

Take EF equal to AB, and through E and F draw (by Prob. 1) EG and FH, making the angles GEF and HFE respectively equal to C and D.

If the angles C and D be together less than two right angles then GEF and HFE will be also together less than

two right angles, and the lines EG and FH will meet if produced far enough on the side of EF on which G and H lie;

let them be produced and meet in K,

then KEF is a triangle having the angles at E and F equal to the angles C and D respectively and the side

EF equal to AB.

There is no limit to the possibility of this construction, except that the sum of the given angles be less than two right angles, as mentioned above.

PROBLEM 5.

Given two straight lines and an angle. To construct a triangle having two of its sides respectively equal to the two given lines, and the angle opposite to one of these sides equal to the given angle.

Let *a* and *b* be the given straight lines, and C the given angle.

It is required to construct a triangle having two of its sides respectively equal to *a* and *b*, and the angle opposite to one of these sides as *a* equal to the angle C.

Take an indefinite straight line DE, and through any point D in DE draw the indefinite straight line DF, making the angle FDE equal to C.

From DF cut off DG equal to *b*, and from G draw GH perpendicular to DE or DE produced.

With centre G and radius equal to *a* describe a circle, this circle will meet DE, or DE produced, in two points K and L equidistant from H, or in the single point H, or in no point whatever, according as *a* is greater than, equal to, or less than GH (Bk. II. Prop. 3).

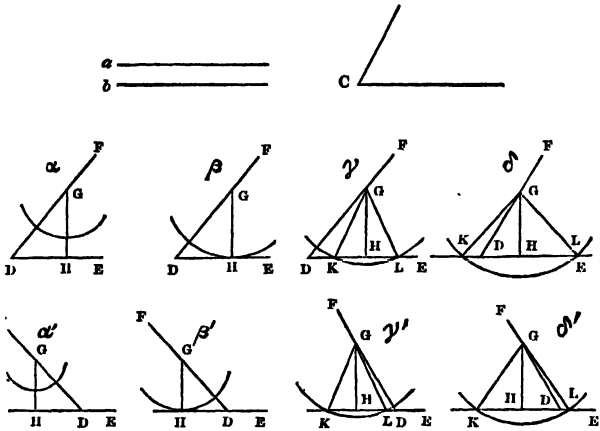
Join GK and GL.

Then one or both of the triangles GDK and GDL on the first supposition, or the triangle GDH on the second supposition, may satisfy the given conditions, and on the third

supposition no triangle whatever can be found satisfying these conditions.

The different cases which may occur are represented in the following eight figures, wherein in α, β, γ, and δ the given angle is *acute*, and in α', β', γ', and δ' this angle is *obtuse*.

Fig. 5.



It is clear that the conditions will be satisfied by the triangle GDH of (β), *either* of the triangles GDK and GDL of (γ), and the triangle GDL alone of (δ) and (δ').

In (γ) GK is less than GD and greater than GH, i.e. a is less than b and greater than GH,

therefore if the given angle be *acute*, and the given side opposite to the given angle be less than the given side containing that angle, there may be two triangles, and there may be none at all.

And, similarly, from (δ) and (δ') we see that, whether the given angle be *acute* or *obtuse*, if the given side opposite to that angle be greater than the given side containing it, there must be one triangle, and only one.

Corollary.—It appears from case (γ) that the triangles

DGK and DGL have the two sides DG and GK of the one equal respectively to the two sides DG and GL of the other, and the angle opposite to the side GK of the one equal to the angle opposite to the side GL of the other; but that the angle opposite to the other side GD is *obtuse* in the one of these triangles and *acute* in the other, therefore

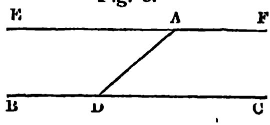
If two triangles have two sides of the one equal to the two sides of the other, each to each, and the angle opposite to one of these sides of the one equal to the angle opposite to the side equal to it of the other, then the two triangles will be in all their parts equal provided the angles opposite to the other two equal sides in each be both acute or both obtuse (Bk. I. page 46).

PROBLEM 6.

Through a given point to draw a straight line parallel to a given straight line.

Let A be the given point, and BC the given straight line. It is required to draw through A a straight line parallel to BC.

Fig. 6.



Take any point D in BC; join AD, and through A draw EA, making the angle EAD equal to the angle ADC, and produce EA to F. Then EF shall be the line required.

Because AD cuts the two straight lines EF and BC, and makes the alternate angles EAD and ADC equal to one another; therefore EF is parallel to BC, and it is drawn through the given point A.

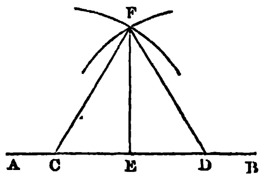
PROBLEM 7.

From a given point in a given straight line to draw a straight line at right angles to the given straight line.

Let AB be the given straight line, and E the given point in it.

It is required to draw from E a straight line at right angles to AB.

Fig. 7.



Take two points C and D in AB, equidistant from E.

With centre C and radius CD describe a circle.

With centre D and radius DC describe a circle.

These circles will cut one another on opposite sides of AB, because the distance between their centres is less than the sum and greater than the difference of their radii.

Let F be the point of intersection above AB, and join EF;

then EF shall be at right angles to AB;

join CF and DF.

Because, in the triangles CEF and DEF, the three sides CE, EF, and CF, are respectively equal to the three sides DE, EF, and DF;

therefore the angle CEF is equal to the angle DEF (Bk. I. Prop. 7);

therefore EF is at right angles to AB.

If E be joined with the point of intersection of the circles *below* AB it may be proved, as above, that the line so drawn will be also at right angles to AB.

PROBLEM 8.

To bisect a given finite straight line.

Let AB be the given finite straight line.

It is required to bisect AB.

With centre A and radius AB describe the circle CBD.

And with centre B and radius BA describe the circle CAD.

These two circles must intersect in two points on opposite sides of AB, because the distance between their centres is

less than the sum and greater than the difference of their radii.

Let these points of intersection be C and D.

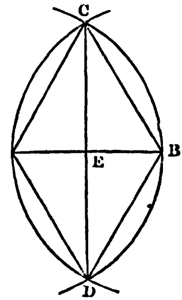
Join CD, and let CD cut AB in E, then E is the point of bisection of AB.

Join AC, CB, BD, and DA.

Because AC is equal to BC, therefore C is a point in the line bisecting AB at right angles. (Bk. I. sec. v. note.)

Similarly, D is a point in the same line,

Fig. 8.



therefore CD bisects AB at right angles (Ax. 2),
therefore AB is bisected in the point E.

PROBLEM 9.

To bisect a given angle.

Let BAC be the given angle.

It is required to bisect BAC.

Take D and E in AB and AC, equally distant from A.

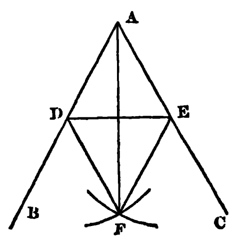
With centre D and radius DE describe a circle.

With centre E and radius DE describe a circle.

Then these two circles must cut one another in two points on opposite sides of the straight line joining D and E. (Bk. II. Props. 16 to 19, *Cor.* 1.)

Let F be the point of intersection on the side of DE remote from A, and join AF, then AF bisects BAC.

Fig. 9.



Join DF and EF.

Because in the two triangles DAF and EAF the sides DA, AF, and DF are respectively equal to EA, AF, and EF, therefore the angle DAF is equal to the angle EAF. (Bk. I. Prop. 7.)

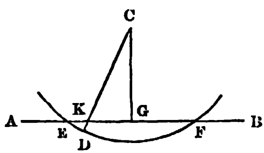
PROBLEM 10.

From a given point outside a given straight line of indefinite length to draw a straight line perpendicular to the given straight line.

Let AB be the given straight line, and C the given point without it.

It is required to draw from C a straight line perpendicular to AB.

Fig. 10.



Take some point D on the side of A B, remote from C, join CD, let CD meet AB in K, and with centre C and radius CD describe a circle.

Because CD is greater than CK, and CK is either the perpendicular from C upon AB or is greater than that perpendicular, therefore the circle described with centre C and radius CD cuts the straight line AB in two points E and F equidistant from the foot of the perpendicular from C upon AB. (Bk. II. Prop. 3.)

Therefore if the middle point G of EF be found by Problem 8, CG will be the perpendicular required.

EXAMPLES.

1. Given the lengths of two sides and one diagonal, required to construct the parallelogram.

2. Given the lengths of one side and two diagonals, required to construct the parallelogram.

3. Given the lengths of the two diagonals and the angle between them, required to construct the parallelogram.

4. Given the lengths of two sides and of the line from one of the angles to the middle point of the opposite side, required to construct the triangle.

5. Given the base, the difference of the two sides, and the difference of the angles at the base, required to construct the triangle.

6. Given the base, the sum of the two sides, and the angle at the vertex, required to construct the triangle.

7. To divide a right angle into three equal parts.

8. To describe a square on a given straight line.

9. From a given point outside a given straight line to draw a straight line, making with the given straight line an angle equal to a given angle.

10. Through a given point to draw a straight line, such that the part of it intercepted between two parallel straight lines may have a given length.

11. Through a given point to draw three straight lines of three given lengths respectively, and such that their extremities may be upon the same straight line, and intercept equal distances upon that straight line.

12. Through a given point to draw a straight line, making equal angles with two given intersecting straight lines.

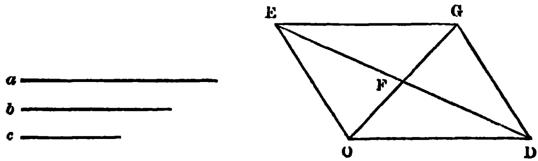
13. Construct a quadrilateral having given the four sides and the straight line joining the middle points of two opposite sides.

14. Draw a straight line of given length so that its extremities may rest on two given straight lines and itself be parallel to a given straight line.

Note.—Solutions are here given of Examples 11 and 12 in illustration of the analytical method referred to above, on page 97.

Ex. 11. Let *a*, *b*, and *c*, be the three given straight lines, and O the given point.

Fig. 11.



Suppose the construction completed, and that OD, OE, and OF are the three lines drawn through the point O as required, equal to *a*, *b*, and *c*, respectively.

Then D, F, and E are in the same straight line, and DF is equal to FE.

If OF be produced to G, so that FG is equal to OF, and EG and GD be joined, then the figure OEGD will be a parallelogram, and the triangle OGD will have its sides equal to *a*, *b*, and 2*c*, respectively,

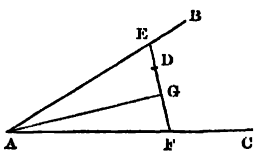
whence this construction.

Construct the triangle OGD, having its sides equal to the three given lines *a*, *b*, and 2*c* (Bk. III. Prob. 2).

Through G draw GE parallel to OD and through O draw OE parallel to GD.

Join DE, and let DE meet OG in F.

Fig. 12.



Then OD, OF, and OE are the straight lines required.*

Ex. 12. Let AB and AC be the two given straight lines and D the given point.

Suppose the construction completed, and that EDF is the straight line passing through D and making equal angles with AB and AC.

* The problem will be impossible if any one of the three lines *a*, *b*, and 2*c*, be greater than the sum of the other two.

Then AEF is an isosceles triangle, and therefore the straight line AG bisecting the angle BAC is perpendicular to EDF,

whence this construction.

Bisect the given angle BAC by the straight line AG.

Through D draw DG at right angles to AG.

Then DG produced both ways will be the line required.

PROBLEM 11.

To find the centre of a given circle or a given arc of a circle.

Let ABC be the given circle (Fig. 13) or given arc of a circle (Fig. 14).

It is required to find the centre of ABC.

Fig. 13.

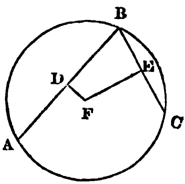
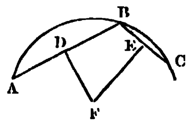


Fig. 14.



Take any three points, A, B, and C, upon ABC, and join them by the straight lines AB and BC.

Bisect AB and BC in D and E, and draw DF and EF through D and E at right angles to AB and BC respectively.

Then from Prop. 5, Bk. II., it follows that F, the point of intersection of DF and EF, is the centre of every circle passing through the points A, B, and C.

Therefore F is the centre of the given circle, or given arc of a circle, ABC.

PROBLEM 12.

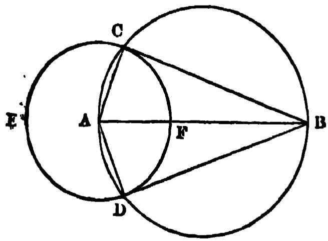
To draw a tangent to a given circle from a given point without it.

Let CED be the given circle, and B the given point with-

out it. It is required to draw from B a straight line touching the circle CED.

Because it is always possible to draw two tangents to the circle CED from the point B (Bk. II. Prop. 20), let BC be one of these.

Fig. 15.



Find the centre A of the circle CED and join AC.

Then ACB is a right angle (Bk. II. Prop. 14).

Therefore if AB be bisected in F, and a circle ACB be described with centre F and radius AF, the point C will lie on the circle ACB (Bk. II. Prop. 22, *Cor.*).

Also, it lies on the circle CED ;

whence this construction.

Find A the centre of the given circle and join AB.

Bisect AB in F.

With centre F and radius FA describe the circle CAD, cutting the circle CED in two points, C and D, and join BC and BD.

Then BC and BD are tangents to the given circle from the point B without it.

Proof.—Because ACB is a semicircle, therefore the angle ACB is a right angle (Bk. II. Prop. 13).

Because AC is the radius to the point C on the circle CDE, and CB is at right angles to AC,

therefore CB is a tangent to the circle CDE at C,

and CB passes through B.

If the given point is on the circle it is only necessary to draw the diameter through this point and at the point to draw a straight line perpendicular to this diameter.

PROBLEM 13.

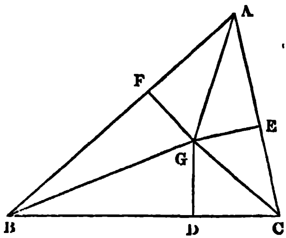
To inscribe a circle in a given triangle.

Let ABC be the given triangle.

It is required to inscribe a circle in ABC.

If possible, let there be a circle touching the sides of the triangle ABC in the points D, E, and F.

Fig. 16.



Find G the centre of this circle, and join GA, GB, and GC.

Because AF and AE are tangents to the circle from the external point A, therefore the angles GAE and GAF are equal;

therefore AG bisects the angle BAC.

Similarly, BG and CG bisect the angles at B and C respectively,

whence this construction.

Bisect the angles at A and B by the straight lines AG and BG; these lines must meet in some point G, because GAB and GBA are together less than two right angles. Then G shall be the centre of the required circle.

Proof.—Draw GD, GE, and GF perpendicular to BC, AC, and AB respectively.

In the triangles GAF and GAE, the angles at A are equal, and the angle at E is equal to the angle at F, each of them being a right angle, and the side AG is common, therefore GF is equal to GE, and similarly GF is equal to GD; therefore GD, GE, and GF are all equal, and the circle described, with G as centre and one of these lines as radius, will pass through the extremities of the other two, and will touch the sides BC, AC and AB in the points D, E, and F respectively since the angles at D, E, and F are right angles.

Corollary.—The three bisectors of the angles of a triangle meet in a point.

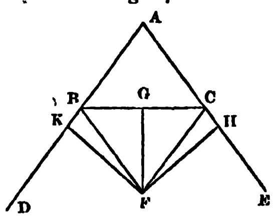
PROBLEM 14.

To draw a circle touching one side of a given triangle externally, and the other two sides produced.

Let ABC be the given triangle.

It is required to draw a circle touching the side BC externally, i.e. on the side remote from A, and the sides AB and AC produced to D and E.

Fig. 17.



If possible, let F be the centre of the required circle and join FB and FC.

Because BD and BC are tangents to the circle drawn from the external point B, therefore FB bisects the angle DBC. (Bk. II. Prop. 20.)

Similarly, FC bisects the angle ECB,

whence this construction.

Bisect the angles DBC and BCE by the lines BF and CF.

BF and CF must meet in some point F, because FBC and FCB are together less than two right angles. Then F shall be the centre of the required circle.

Proof.—From F draw FG, FH, and FK perpendicular to BC, AE, and AD respectively.

It may be proved, as in the last proposition, that the circle described with centre F, and radius equal to either of these perpendiculars, touches the line BC externally and the sides AB and AC produced.

Corollary.—AF would bisect the angle BAC, and therefore the lines bisecting one interior angle and the angles exterior to the two remaining angles of a triangle meet in a point.

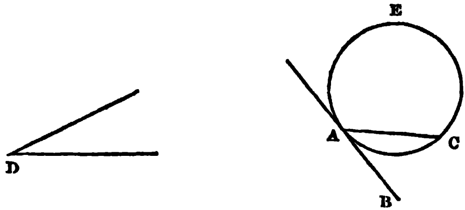
PROBLEM 15.

To draw a straight line, cutting off from a given circle a segment containing an angle equal to a given angle.

Let ACE be the given circle and D the given angle. It

is required to draw a straight line, cutting off from the circle ACE a segment containing an angle equal to D.

Fig. 18.



Suppose AC to be the line required, so that the segment AEC contains an angle equal to D.

Draw AB touching the circle at A.

Then (Bk. II. Prop. 22) the angle BAC is equal to the angle in the segment AEC, and is therefore equal to D,

whence this construction.

Take any point A on the circumference of the circle ACE, and through A draw AB touching the circle (Prob. 13). Through the point A draw the straight line AC making the angle BAC equal to D.

Then AC is a line satisfying the required condition. (Bk. II. Prop. 21.)

Note.—The method of solution adopted explicitly in Problem 8 and in Problems 11 to 13, and *implied* more or less in all the problems of this book, is called that of *the intersection of loci*, and is of great use in the solution of problems of construction.

Thus in Problem 12 we *prove* that the required point must be one of the points of the locus defined by the circle BCD upon AB as diameter.

We know also that in this case the point C must be upon the circle CDE.

Therefore the points in which the two circles BCD and CDE intersect must be positions of the point required.

So again in Problem 8, we know that every point equi-

distant from A and B must be situated on the straight line bisecting AB at right angles.

Therefore C and D must be situated on the straight line bisecting AB at right angles,

therefore the straight line CD bisects AB at right angles,

therefore every point equidistant from A and B must be on the straight line CD,

therefore E, the intersection of AB and CD, bisects AB.

EXAMPLES.

1. To describe a circle passing through two given points and having a given radius.

2. To describe a circle passing through one given point, having a given radius, and having its centre upon a given straight line.

3. Describe a circle passing through two given points and having its centre on a given straight line, or on the circumference of a given circle.

4. Describe a circle, with given radius, passing through a given point, and touching a given straight line or a given circumference.

5. Describe a circle passing through a given point and touching two given straight lines, one of them at any point whatever, and the other in a certain given point upon it.

6. Describe a circle touching a given straight line, and touching a given circumference in a given point.

7. Given two circles, such that one does not lie entirely within the other, to draw a common tangent to these two circles.

8. Through one of the points of intersection of two given circles to draw a straight line, such that the parts of this line intercepted by the two circles may be of equal length.

9. Given two concentric circles, draw a straight line which shall be divided into three equal parts in its points of intersection with the two circles.

BOOK IV.

ON AREAS.

Section I.—Theorems.

DEFINITIONS.

44.—The *area* of a limited surface is its extent or magnitude.

45.—Two plane figures are said to have *equal areas*—

1st. When they can be superposed one upon the other in such a way that their boundaries coincide throughout.

2nd. When they can be divided into the same number of parts, such that to every part of one of the figures there is a corresponding part of the second figure which can be superposed, as in case 1, upon the corresponding part of the first figure.

Fig. 1.

Fig. 2.

Fig. 3.

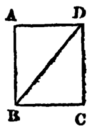
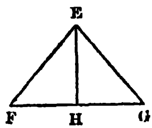


Fig. 4.



Thus, if the triangle ABC can be superposed upon the triangle DEF in such a manner that the boundaries of the two triangles coincide throughout, then the area of ABC is equal to that of DEF.

Or again, if the triangles BAD and DCB, which make up the parallelogram ABCD, can be superposed upon the triangles EHF and EHG, which make up the triangle EFG, in such a manner that the boundaries of BAD and DCB coincide throughout with the boundaries of EHF and EHG respectively, then the area of ABCD is equal to the area of EFG.

DEFINITIONS.

46.—When one side of a triangle, or of a parallelogram, is called the *base*, the perpendicular upon this side from the opposite angle of the triangle, or from any point in the opposite side of the parallelogram, is called the *altitude*.

47.—Triangles and parallelograms are *between the same parallels* when their bases are in the same straight line, and when the line joining the vertices of the triangles, in the one case, or the sides of the parallelograms opposite to the bases, in the other case, are in a straight line parallel to the bases.

Fig. 5.

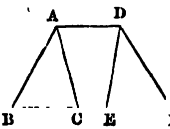
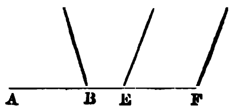


Fig. 6.

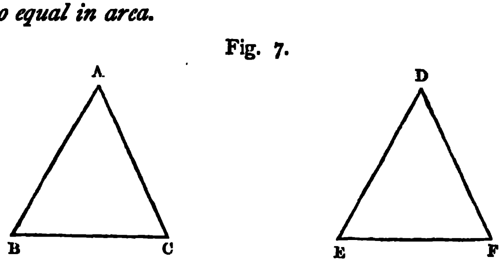


Thus if AD and BF be parallel in Fig. 5, or if DCHG be one straight line in Fig. 6, the triangles ABC and DEF, or the parallelograms BD and EG, are between the same parallels.*

PROPOSITION I.

Triangles which are equal to each other in all their parts are also equal in area.

Fig. 7.



* A parallelogram is often denoted by the letters at a pair of opposite angular points; thus, the parallelogram ABCD (Fig. 6) is called the parallelogram DB.

Let ABC and DEF be two triangles equal to each other in all their parts, then the area of the triangle ABC shall be equal to the area of the triangle DEF.

Because the triangles ABC and DEF are equal to each other in all their parts, therefore the triangle ABC can be applied to the triangle DEF in such a way that each angular point and side of the one triangle coincides respectively with an angular point and side of the other triangle (Bk. I. Prop. 5, *Cor.* 2).

Therefore the boundaries of the two figures may be made to coincide throughout.

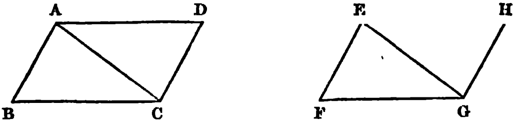
Therefore the area of the triangle ABC is equal to the area of the triangle DEF.

Corollary.—It follows from this Proposition, and Prop. 25, Bk. I., that the two triangles into which a parallelogram is divided by either diagonal are equal to each other in area, and therefore that the area of the parallelogram is double of the area of either of these triangles.

PROPOSITION 2.

Two parallelograms which have two adjacent sides and the included angle of the one respectively equal to two adjacent sides and the included angle of the other are also equal in area.

Fig. 8.



Let ABCD and EFGH be two parallelograms which have the adjacent sides AB and BC and the included angle ABC of the one equal respectively to the adjacent sides EF and FG and the included angle EFG of the other, then shall the area of the parallelogram BD be equal to the area of the parallelogram FH.

Join AC and EG.

Because AB, BC and the angle ABC are respectively equal to EF, FG and the angle EFG, therefore the triangles ABC and EFG are equal to each other in all their parts.

Therefore the area of the triangle ABC is equal to the area of the triangle EFG (Bk. IV. Prop. 1).

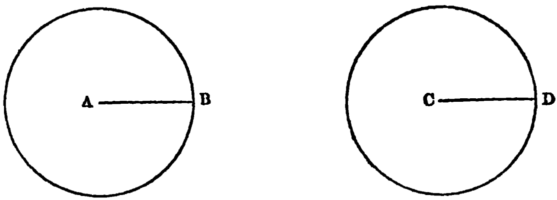
But the areas of the parallelograms BD and FH are respectively double of the areas of the triangles ABC and EFG (Bk. IV. Prop. 1, *Cor.*).

Therefore the area of the parallelogram BD is equal to the area of the parallelogram FH.

PROPOSITION 3.

Circles which have equal radii are also equal in area.

Fig. 9.



Let A and C be the centres of two circles whose radii AB and CD are equal, then shall the area of the circle A be equal to the area of the circle C.

Apply the circle A to the circle C, so that the points A and C coincide.

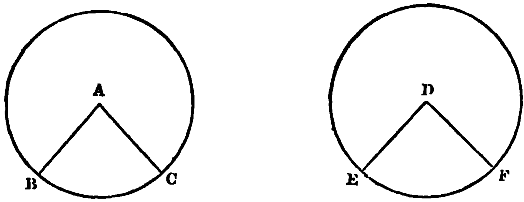
Because the centres of the circles A and C coincide, and their radii are equal, therefore the circumferences of the circles coincide in every point (Bk. II. Prop. 2).

Therefore the area of the circle A is equal to the area of the circle C (Def. 45).

Corollary.—By similar reasoning it may be shown that in

equal circles the sectors corresponding to equal angles are equal to one another in area,

Fig. 10.



or that if A and D be equal circles, and BAC and EDF be equal angles then the areas of the sectors BAC and EDF will be also equal.

PROPOSITION 4.

Parallelograms upon the same base and between the same parallels are equal to one another in area.

Fig 11.

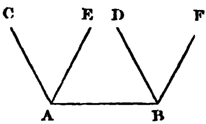
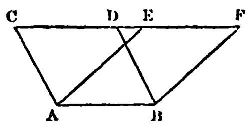


Fig. 12.



Let the two parallelograms CABD and EABF be upon the same base AB and between the same parallels AB and CF, then the area of the parallelogram AD shall be equal to the area of the parallelogram AF (Figs. 11 and 12).

Because AC and BD are opposite sides of a parallelogram,
therefore AC is equal to BD.

Because CF meets the parallels AC and BD;
therefore the angle ECA is equal to the angle FDB.
(Bk. I. Prop. 19.)

Similarly the angle CEA is equal to the angle DFB;

therefore the triangles ECA and FDB are equal in all their parts (Bk. I. Prop. 23, *Cor.* and *Note*);

therefore the area of the triangle ECA is equal to the area of the triangle FDB;

therefore the difference of the areas of the figure CABF and the triangle ACE is equal to the difference of the areas of the same figure CABF and the triangle BDF.

But the difference of the areas of CABF and ACE is the area of the parallelogram AF,

and the difference of the areas of CABF and BDF is the area of the parallelogram AD,

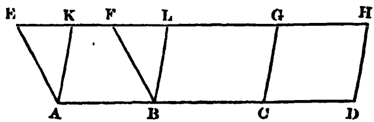
therefore the area of the parallelogram AF is equal to the area of the parallelogram AD.

PROPOSITION 5.

Parallelograms upon equal bases and between the same parallels are equal to one another in area.

Let ABFE and CDHG be two parallelograms upon equal bases AB and CD and between the same parallels AD and EH, then shall the area of the parallelogram AF be equal to the area of the parallelogram CH.

Fig. 13.



Through the points A and B, draw AK and BL parallel to CG or DH, so that each of the figures AKLB and AKGC is a parallelogram.

Because the parallelograms AF and BK are upon the same base and between the same parallels, therefore their areas are equal (Bk. IV. Prop. 4).

Because the straight line AD meets the parallels AK and CG, therefore the angle KAB is equal to the angle GCD (Bk. I. Prop. 19).

Because AKGC is a parallelogram, therefore AK is equal to CG (Bk. I. Prop. 25).

Because the sides KA and AB and the included angle KAB of the parallelogram KB are respectively equal to the sides GC and CD and the included angle GCD of the parallelogram GD, therefore the area of KB is equal to the area of GD (Bk. IV. Prop. 2).

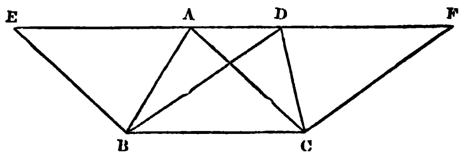
But the area of KB has been proved equal to the area of AF,

therefore the area of AF is equal to the area of GD.

PROPOSITION 6.

Triangles upon the same base and between the same parallels are equal to one another in area.

Fig. 14.



Let the triangles ABC and DBC be upon the same base BC, and between the same parallels BC and AD, then shall the area of the triangle ABC be equal to the area of the triangle DBC.

Through B and C draw BE and CF parallel to AC and BD respectively, and meeting AD produced in the points E and F respectively.

Then the figures ACBE and DBCF are parallelograms, and their areas are double of the areas of the triangles ABC and DBC respectively.

Because the parallelograms CE and BF are upon the same base and between the same parallels, therefore the area of CE is equal to the area of BF;

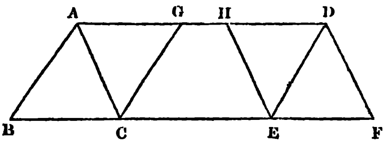
therefore half the area of CE is equal to half the area of BF;
therefore the area of the triangle ABC is equal to the area of the triangle DBC (Bk. IV. Prop. 1, *Cor.*).

PROPOSITION 7.

Triangles upon equal bases and between the same parallels are equal to one another in area.

Let ABC and DEF be two triangles upon the equal bases BC and EF, and between the same parallels BF and AD, then shall the area of the triangle ABC be equal to the area of the triangle DEF.

Fig. 15.



Through the points C and E draw CG and EH respectively parallel to AB and DF, then the figures ABCG and DFEH are parallelograms, and their areas are double of the areas of the triangles ABC and DFE respectively.

Because the parallelograms BG and FH are upon equal bases and between the same parallels, therefore the area of BG is equal to the area of FH;
therefore half the area of BG is equal to half the area of FH;
therefore the area of the triangle ABC is equal to the area of the triangle DEF (Bk. IV. Prop. 1, *Cor.*).

N.B.—When two figures have equal areas they are frequently said to be *equal* to one another.

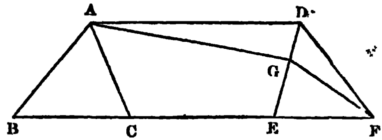
PROPOSITION 8.

Equal triangles upon equal bases in the same straight line, and on the same side of that line, are between the same parallels.

Let the two equal triangles ABC and DEF be upon the

equal bases BC and EF, in the same straight line BF, and let them be situated on the same side of BF, then if AD be joined AD shall be parallel to BF.

Fig. 16.



If AD be not parallel to BF draw through A the line AG parallel to BF, meeting ED, or ED produced, in the point G, and join GF.

Because the triangles ABC and GEF are upon equal bases and between the same parallels, therefore the area of ABC is equal to the area of GEF.

But the area of ABC is equal to the area of DEF,

therefore the area of DEF is equal to the area of GEF,

which is impossible;

therefore no line but AD can be parallel to BF,

that is, AD is parallel to BF.

It may be proved by demonstration in all respects similar to that in the proposition that—

Corollary 1.—Equal parallelograms upon equal bases in the same straight line and on the same side of that line are between the same parallels.

Corollary 2.—Equal triangles upon the same base and on the same side of it are between the same parallels.

Corollary 3.—Equal parallelograms upon the same base and upon the same side of it are between the same parallels.

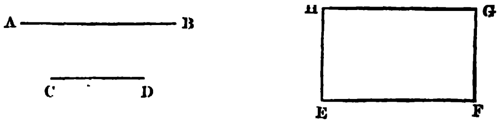
DEFINITIONS.

48.—If a rectangle be constructed having its adjacent sides respectively equal to two given straight lines, the area of this rectangle is called the *rectangle contained by these two straight lines*, or the *rectangle of these two straight lines.*

Thus, if AB and CD be two straight lines, and EFGH be a rectangle whose adjacent sides EF and EH are respectively equal to AB and CD, then the area of the rectangle

EFGH is called the *rectangle of AB and CD*, or *the rectangle contained by AB and CD.*

Fig. 17.



This is written either *the rectangle* AB . CD or AB . CD *simply.*

49.—If CD were equal to AB the area of the corresponding rectangle would be called the *square upon* AB, or the *square of* AB, and this would be written either *sq. of* AB or AB^2.

N.B.—The term square of a line is sometimes employed to denote the *figure* and sometimes the *area* of the figure.

PROPOSITION 9.

The area of a parallelogram is equal to the rectangle of its base and altitude.

Fig. 18.

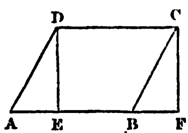
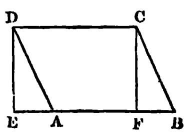


Fig. 19.



Let ABCD be a parallelogram, and let DE be the perpendicular dropped from D upon AB or AB produced, then shall the area of the parallelogram ABCD be equal to the rectangle of AB and DE.

Complete the rectangular parallelogram DEFC.

Because the parallelograms AC and EC are upon the same base DC, and between the same parallels, therefore the area of AC is equal to the area of EC.

But the area of EC is the rectangle of DE and DC, that is, of DE and AB, since AB is equal to DC,

therefore the area of the parallelogram ABCD is equal to the rectangle of DE and AB.

PROPOSITION 10.

The area of a triangle is equal to half the rectangle of its base and altitude.

Fig. 20.

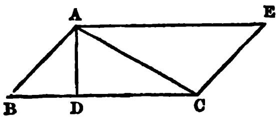
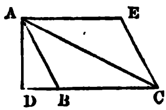


Fig. 21.



Let ABC be a triangle, and let AD be the perpendicular dropped from A upon BC, Fig. 20, or BC produced, Fig. 21, then shall the area of ABC be equal to half the rectangle of BC and AD.

Through A and C (Figs. 20 and 21) draw AE and CE parallel to BC and BA respectively, thus completing the parallelogram ABCE.

Because BE is a parallelogram, and AC its diameter, therefore the area of the triangle ABC is half the area of the parallelogram BE.

But the area of BE is the rectangle of BC and AD (Bk. IV. Prop. 9);

therefore the area of ABC is half the rectangle of BC and AD.

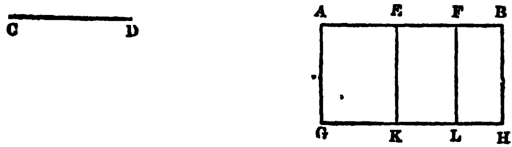
Corollary.—If a parallelogram and a triangle be upon the same base and between the same parallels, the area of the triangle will be half the area of the parallelogram.

PROPOSITION 11.

If there be two straight lines, one of which is divided into any number of parts, the rectangle of the two straight lines

shall be equal to the sum of the rectangles of the undivided line, and of the several parts of the divided line.

Fig. 22.



Let AB and CD be two straight lines, and let AB be divided into three parts in the points E and F, then shall the rectangle of CD and AB be equal to the sum of the rectangles of CD and AE, CD and EF, and CD and FB.

Through A draw AG at right angles to AB and equal to CD, and complete the rectangle AGHB.

Through E and F draw EK and FL, at right angles to AB, and meeting GH in K and L.

Then EK and FL are parallel to AG and to one another (Bk. I. Prop. 18, *Cor.*), and therefore AK, EL and FH are rectangles, and each of the lines EK and FL is equal to AG, that is, to CD (Bk. I. Prop. 25).

Now the area of AH is equal to the sum of the areas of AK, EL, and FH.

But the area of AH is the rectangle of AG and AB, that is, of CD and AB.

Similarly, the areas of AK, EL, and FH are the rectangles of CD and AE, CD and EF, and CD and FB.

therefore the rectangle of CD and AB is equal to the sum of the rectangles of CD and AE, CD and EF, and CD and FB.

If the number of parts were greater or less than three, a similar proof would apply in all respects.

Corollary 1.—If CD were equal to AB, then the rectangle of AB and CD would be equal to the square of AB, and the sum of the rectangles of CD, and each part of AB,

would be equal to the sum of the rectangles of AB, and each part of AB, therefore

If a straight line be divided into any number of parts, the square of the whole line will be equal to the sum of the rectangles of the whole line, and each of the parts.

Corollary 2.—If AB were divided into *two* parts only, as AE and EB, and if CD were equal to EB, then the rectangle of AB and CD would be equal to the rectangle of

Fig 23.

AB and the part EB, also the rectangle of CD and AE would be equal to the rectangle of the parts AE and EB, and the rectangle of CD and EB would be equal to the square of EB, therefore

If a straight line be divided into any two parts, the rectangle of the whole line and one of the parts will be equal to the rectangle of the two parts, together with the square of the aforesaid part.

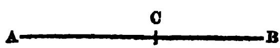
Corollary 3.—If AB were divided into *two equal* parts in E, then the rectangle of CD and AB would be equal to the rectangle of CD and twice AE, also the sum of the rectangles of CD and AE, and of CD and EB would be equal to twice the rectangle of CD and AE, therefore

The rectangle of any two straight lines is equal to twice the rectangle of one of the lines and half the other line.

PROPOSITION 12.

If a straight line be divided into any two parts, the square of the whole line shall be equal to the sum of the squares of the two parts, together with twice the rectangle of those parts.

Fig. 24.



Let AB be divided into two parts in C, then shall the square of AB be equal to the sum of

the squares of AC and CB, together with twice the rectangle of AC and CB.

Because AB is divided into two parts in C,

therefore AB^2 is equal to AB . AC+AB . CB (Bk. IV. Prop. 11, *Cor.* 1).

But AB . AC is equal to AC . CB + AC^2 (Bk. IV. Prop. 11, *Cor.* 2),

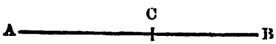
and AB . CB is equal to AC . CB+CB^2 (Bk. IV. Prop. 11, *Cor.* 2);

therefore AB^2 is equal to AC^2+CB^2+2 . AC . CB.

PROPOSITION 13.

If a straight line be divided into any two parts, the square of one of the parts shall be less than the sum of the squares of the whole line, and of the other part by twice the rectangle of the whole line and that other part.

Fig. 25.



Let the straight line AB be divided into any two parts in the point C, then the square of AC shall be less than the sum of the squares of AB and CB by twice the rectangle of AB and CB.

Because AB is divided into any two parts in the point C,

therefore AB^2 is equal to AC^2+CB^2+2AC . CB (Bk. IV. Prop. 12).

To each of these equals add the square of CB,

therefore AB^2+CB^2 is equal to AC^2+2CB^2+2AC . CB.

Again, because AB is divided into two parts in the point C,

therefore AB . BC is equal to AC . CB + CB^2 (Bk. IV. Prop. 11, *Cor.* 2),

therefore 2 . CB^2 + 2 . AC . CB is equal to 2 . AB . BC,

therefore AB^2+CB^2 is equal to AC^2+2 . AB . BC,

therefore the square of AC is less than the sum of the squares of AB and BC by twice the rectangle of AB and BC.

Note.—The last two propositions may be thus expressed respectively.

1st. The square of the sum of two straight lines is greater than the sum of the squares of these lines, by twice the rectangle contained by these lines.

2nd. The square of the difference of two straight lines is less than the sum of the squares of these lines by twice the rectangle contained by these lines.

Note.—Hitherto we have denoted the difference between two magnitudes by placing the sign $\sim$ between them; but when one of the two magnitudes is *known* to be greater than the other, this difference is frequently denoted by means of the sign $-$ instead of the sign $\sim$, the greater of the two magnitudes preceding and the smaller following the sign $-$.

PROPOSITION 14.

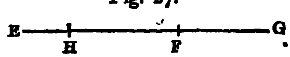
The difference of the squares of two straight lines is equal to the rectangle of the sum and difference of these lines.

Fig. 26.

Let AB and CD be two straight lines, of which AB is the greater; then shall the difference of the squares of AB and CD be equal to the rectangle of the sum and difference of AB and CD.

Let the straight lines EF and FG be equal to AB and CD respectively, and let them be placed in the same straight line with a common extremity at F.

Fig. 27.



From EF cut off FH equal to CD.

then EG is the sum and EH the difference of AB and CD,

Because EF is divided into two parts in the point H,

therefore EF^2 is equal to $EH^2 + HF^2 + 2 \cdot EH \cdot HF$,

take away HF^2 from each of these equals,

therefore $EF^2 - HF^2$ is equal to $EH^2 + 2 \cdot EH \cdot HF$.

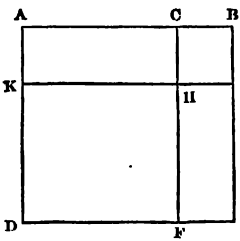
therefore $EF^2 - HF^2$ is equal to $EH^2 + EH . HG$ (Bk. IV. Prop. 11, *Cor.* 3),

therefore $EF^2 - HF^2$ is equal to $EH . EG$ (Bk. IV. Prop. 11, *Cor.* 2).

Note.—The following independent proofs of the last three propositions, will afford an instructive exercise to the reader.*

Let AB be a straight line divided into two parts in C. Upon AB describe the square ADEB; through C draw CHF parallel to AD and BE; mark off BG upon BE, equal to BC, and through G draw GHK parallel to AB or DE (Fig. 28).

Fig. 28.



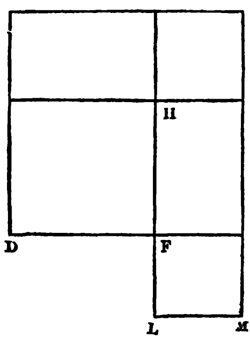
From the construction it results that the four figures AH, HE, BH, and HD are parallelograms, and also that BH and HD are squares, they are therefore the squares of BC and KH, i. e. of BC and AC.

Also AH is the rectangle of AC and CH, i.e. of AC and CB.

Similarly, HE is the rectangle of HG and GE, i.e. of CB and AC,

therefore the square of AB is equal to the squares of AC and CB, together with twice the rectangle of AC and CB.

Fig. 29.



Again, make the same construction as in the last case, and in addition describe upon FE the square FM (Fig. 29).

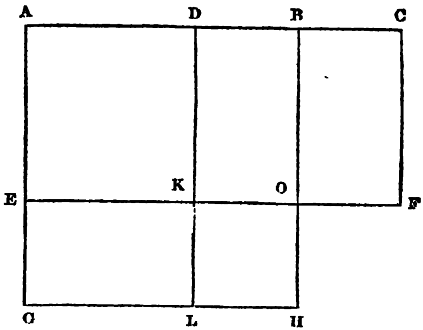
Then AG and GL are, each of them, the rectangle of AB and BC, for GM is equal to GE+EM, i.e. to GE+GB or AB.

* These demonstrations are taken from Wilson's *Geometry.*

Also the figure ABMLFD is made up of the two figures AE and FM, i.e. of the two squares of AB and BC.

But KF, the square of AC, is less than the area of this figure by the sum of the areas of AG and GL, i.e. by twice the rectangle of AB and BC,

Fig. 30.



therefore the square of AC is less than the sum of the squares of AB and BC by twice the rectangle of AB and BC.

Finally, let AB and BC be two straight lines, of which AB is the greater, and let BD be equal to BC; then AC is the sum and AD the difference of AB and BC (Fig. 30).

Upon AB describe the square AH, make AE equal to AD and complete the construction as in the figure, then KH is the square of DB, i. e. the square of BC.

The areas of the figures EL, DO, and BF are all equal, for they are rectangular parallelograms having their adjacent sides equal, each to each.

Also, the difference of the squares of AB and BC is the difference of the areas of AH and KH, and therefore is made up of the areas of the figures AO and EL, i. e. of AO and BF.

But the areas of AO and BF are together equal to the

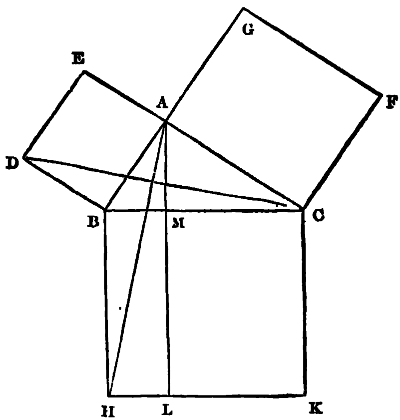
area of AF, i. e. to the rectangle of the sum and difference of AB and BC.

PROPOSITION 15.

In every right-angled triangle, the square of the hypothenuse is equal to the sum of the squares of the other two sides.

Let ABC be a right-angled triangle, of which A is the right angle, then shall the square of BC be equal to the sum of the squares of AB and AC.

Fig. 31.



Produce CA and BA to E and G respectively, so that AE may be equal to AB, and AG to AC, then each of the angles BAE and CAG will be a right angle, because the angle BAC, adjacent to each of them, is a right angle.

Complete the parallelograms AD and AF, which will be the squares described on AB and AC respectively, and upon BC describe the square BHKC.

Join AH and DC, and through A draw AL parallel to BH.

Because each of the angles DBA and CBH is a right angle, therefore DBA is equal to CBH,

therefore DBA + ABC is equal to CBH + ABC,

that is DBC is equal to ABH.

Because AB, BH, and the angle ABH are equal to DB, BC, and the angle DBC respectively,

therefore the triangles ABH and DBC are equal in all their parts,

therefore the area of ABH is equal to the area of DBC.

Because the triangle ABH, and the parallelogram BL are upon the same base and between the same parallels, therefore the area of BL is double of the area of ABH.

Similarly, the area of AD is double of the area of DBC.

But the area of ABH has been proved to be equal to the area of DBC,

therefore the area of AD is equal to the area of BL.

Similarly, the area of AF is equal to the area of CL,

therefore the area of AD + the area of AF is equal to the area of BL + the area of CL, i.e. to the area of BK,

therefore the sum of the squares of AB and AC is equal to the square of BC.

Corollary.—In every right-angled triangle the square of either side is equal to the difference of the squares of the hypothenuse and the remaining side.

DEFINITION.

50.—If perpendiculars be dropped upon a given straight line of indefinite length from each extremity of a given finite

Fig. 32.

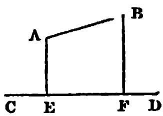


Fig. 33.

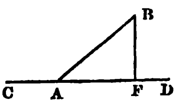
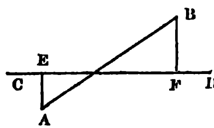


Fig. 34.



straight line, the length of the portion of the first line inter-

cepted between the feet of these perpendiculars is called the projection of the second line upon the first line.

Thus, if AE and BF (Figs. 32 and 34) and BF (Fig. 33) be perpendicular to CD, then EF (Figs. 32 and 34) and AF (Fig. 33) will be the projection of AB upon CD.

PROPOSITION 16.

In any triangle the square of any one side is less *or* greater *than the sum of the squares of the two remaining sides according as the angle included by these two remaining sides is* less *or* greater *than a right angle, and the difference between the square of any side of a triangle and the sum of the squares of the two remaining sides is equal to twice the rectangle contained by either one of these sides and the projection of the other side upon the aforesaid side.*

Fig. 35.

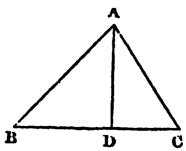
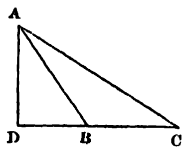


Fig. 36.



Let ABC be any triangle, then the square of any side, as AC, shall be less or greater than the sum of the squares of the two remaining sides AB and BC, according as ABC is less or greater than a right angle, and if AD be the perpendicular from A upon BC, or BC produced, the difference of the square of AC and the sum of the squares of AB and BC shall be equal to twice the rectangle of CB and BD.

1st. Let ABC be less than a right angle, then because ADC is a right angle, therefore ADC must be greater than ABC, and therefore AD must fall *within* the triangle ABC, as in Fig. 35.

Because BC is divided into two parts in D,

therefore DC^2 is equal to $BC^2+BD^2-2CB \cdot BD$

(Bk. IV. Prop. 13).

To each of these equals add the square of AD,

therefore DC^2+AD^2 is equal to $BC^2+BD^2+AD^2 - 2CB \cdot BD$.

Because the angles CDA and BDA are each of them right angles,

therefore DC^2+AD^2 is equal to AC^2;

and BD^2+AD^2 is equal to AB^2,

therefore AC^2 is equal to $AB^2+BC^2-2CB \cdot BD$.

2nd. Let ABC be greater than a right angle, then it may be proved, as in the last case, that AD falls *without* the triangle ABC, as in Fig. 36.

Because DC is divided into two parts in B,

therefore DC^2 is equal to $DB^2+BC^2+2DB \cdot BC$ (Bk. IV. Prop. 12).

To each of these equals add AD^2,

therefore AD^2+DC^2 is equal to $AD^2+DB^2+BC^2 + 2DB \cdot BC$.

Because ADB is a right angle,

therefore AD^2+DC^2 is equal to AC^2;

and AD^2+DB^2 is equal to AB^2,

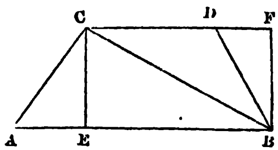
therefore AC^2 is equal to $AB^2+BC^2+2DB \cdot BC$.

PROPOSITION 17.

The area of a trapezoid is equal to half the rectangle of the sum of the two parallel sides and the perpendicular distance between them.

Let ACDB be a trapezoid, of which AB and CD are the two parallel sides, and CE the perpendicular drawn from C to AB, then shall the area of the figure ACDB be equal to half the rectangle of CE and the sum of AB and CD.

Fig. 37.



Join BC, and draw BF perpendicular to CD produced.

The area of the trapezoid ACDB is equal to the sum of the areas of the triangles ABC and BDC.

But the area of the triangle ABC is half the rectangle of AB and CE (Bk. IV. Prop. 10).

Similarly, the area of the triangle BDC is half the rectangle of CD and BF, i.e of CD and CE,

therefore the area of the trapezoid ACDB is equal to half the sum of the rectangles AB . CE and CD . CE.

But the sum of the rectangles AB . CE and CD . CE is equal to the rectangle of CE and the sum of AB and CD (Bk. IV. Prop. 11),

therefore the area of the trapezoid is equal to half the rectangle of CE and the sum of AB and CD.

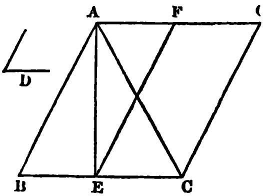
SECTION II.—PROBLEMS OF CONSTRUCTION CONNECTED WITH AREAS.

PROBLEM 1.

Given a triangle and an angle: it is required to construct a parallelogram whose area shall be equal to the area of the triangle, and one of whose angles shall be equal to the given angle.

Let ABC be the given triangle, and D the given angle; it is required to construct a parallelogram whose area shall be equal to the area of ABC, and one of whose angles shall be equal to D.

Fig. 38.



Bisect BC in E (Bk. III. Prob. 8).

Through E draw EF, making the angle FEC equal to D (Bk. III. Prob. 1).

Through A draw AFG parallel to BC, meeting EF in F (Bk. III. Prob. 6).

Through C draw CG parallel to EF, meeting AFG in G.

Then FECG shall be the parallelogram required.

Because the triangles ABE and AEC are upon equal bases and of equal altitudes, therefore their areas are equal (Bk. IV. Prop. 7);

therefore the area of the triangle ABC is double of the area of the triangle AEC.

Because the parallelogram EG and the triangle AEC are upon the same base EC and between the same parallels,

therefore the area of EG is double of the area of AEC,

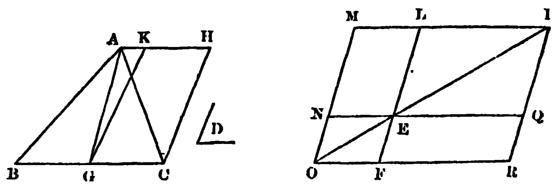
therefore the parallelogram EG is equal in area to the triangle ABC, and it has the angle FEC equal to the angle D.

PROBLEM 2.

Given a triangle, an angle, and a finite straight line: it is required to construct a parallelogram having its area equal to that of the triangle, one angle equal to the given angle, and given straight line for one of its sides.

Let ABC, D, and EF be the given triangle, angle, and straight line respectively.

Fig. 39.



By the last proposition construct the parallelogram CK equal in area to ABC and having the angle KGC equal to D.

Produce FE to L so that EL may be equal to GK, make the angle NEL equal to GCH (Bk. III. Prob. 1), and complete the parallelogram EM (Bk. III. Prob. 6).

Draw FO parallel to EN meeting MN in O and join OE.

Because ML is parallel to and OE meets NE;

therefore ML and OE will meet if produced,

let them be produced till they meet in P.

Draw PR parallel to EL, and let it meet NE and OF produced in Q and R, then ER shall be the parallelogram required.

Because (by Bk. IV. Prop. 1, *Cor.*) the areas of ONE and ELP are respectively equal to the areas of OFE and EQP, and the area of OMP to that of ORP, it follows that the area of EM is equal to that of ER.

Also because the angles NEL and FEQ are equal, the parallelograms EM and ER are equiangular.

Therefore the parallelogram ER is equal in area to, and equiangular with, the parallelogram EM, and it has the straight line EF for one of its sides.

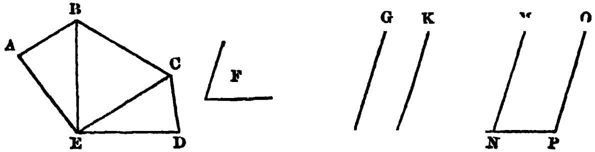
But the parallelogram EM is equal in area to, and equiangular with, the parallelogram CK.

Therefore the parallelogram ER is equal in area to the given triangle ABC, and has an angle equal to the given angle, and the given straight line for one of its sides.

PROBLEM 3.

Given any rectilinear figure and an angle: it is required to construct a parallelogram having its area equal to that of the given figure, and the given angle for one of its angles.

Fig. 40.



Let ABCDE be the given rectilinear figure and F the given angle; it is required to construct a parallelogram

whose area shall be equal to the area of ABCDE, and one of whose angles shall be equal to F.

Divide the figure ABCDE into triangles by the lines BE, CE, &c.

Construct the parallelogram GL equal in area to the triangle ABE, and having the angle GHL equal to F (Bk. IV. Prob. 1).

To KL apply the parallelogram KN equal in area to the triangle BEC, and having the angle KLN equal to F (Bk. IV. Prob. 2).

And repeat the last step for each remaining triangle.

Then the figure GHPO so formed shall be the parallelogram required.

Because the angles GHL and KLN are each equal to F, therefore they are equal to each other.

But GHL and KLH are together equal to two right angles (Bk. I. Prop. 19).

therefore KLN and KLH are together equal to two right angles, and therefore HL and LN are in the same straight line.

Similarly LN and NP are in the same straight line, and so on; therefore HP is a straight line, and similarly GO is a straight line, and HP and GO are parallel straight lines, because the parts of them GK and HL are parallel..

Therefore GP is a parallelogram having an angle GHP equal to F and having its area equal to that of the rectilinear figure ABCDE, because the parts of GP are respectively equal in area to the parts of ABCDE.

PROBLEM 4.

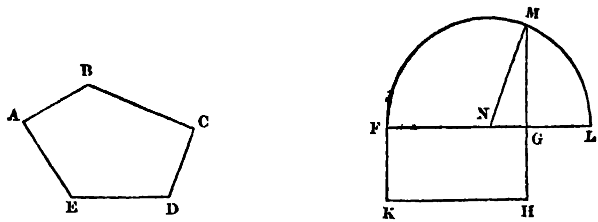
Given a rectilinear figure: it is required to construct a square equal in area to the given rectilinear figure.

Let ABCDE be the given rectilinear figure; it is required to construct a square equal in area to ABCDE.

Construct the rectangular parallelogram FH equal in area to ABCDE by the last Problem.

If FG and GH be equal, FH is a square, and the problem is complete.

Fig. 41.



If not, produce FG to L, make GL equal to GH, upon FL as diameter, describe the semicircle FML, and produce HG to M.

Then the square of GM shall be the square required.

Because FN is equal to NL, therefore FG is equal to the sum of NL and NG, and GL is also equal to their difference;

therefore the rectangle of FG and GL is equal to the difference of the squares of NL and NG (Bk. IV. Prop. 14).

But NL is equal to NM,

therefore the difference of the squares of NL and NG is equal to the difference of the squares of NM and NG, i.e. to the square of GM (Bk. IV. Prop. 15).

And the rectangle of FG and GL is the area of the figure FH, i.e. the area of the given rectilinear figure ABCDE,

therefore the square of GM is equal in area to the rectilinear figure ABCDE.

EXAMPLES.

1. From any point within an equilateral polygon perpendiculars are drawn to the sides, produced if necessary. Prove that if a rectangle of equal area with the polygon be constructed on one of the sides of the polygon, the sum of

these perpendiculars will be equal to twice the side of the rectangle adjacent to the side of the polygon on which it is constructed.

2. If ABC be a triangle right-angled at C, and from AB there be cut off AE equal to AC and BD equal to BC, prove that the square of DE will be equal to twice the rectangle of AD and EB.

3. From one of the extremities A of the base AB of the isosceles triangle CAB, AD is drawn perpendicular to the opposite side BC, meeting that side in D. Prove that the square of AD is equal to the square of BD together with twice the rectangle of BD and DC.

4. If ABC be a triangle right-angled at C, and CD be drawn perpendicular to AB and meeting AB in D, prove that the rectangle of AD and DB is equal to the square of CD.

5. The straight line AB is divided in C so that the rectangle AB . BC is equal to the square of AC. Prove that if CD be cut off from CA equal to BC, then CA will be divided in D so that the rectangle CA, AD is equal to the square of CD.

6. From the extremities A and B, of the base of a triangle ABC, perpendiculars AL and BM are drawn to the opposite sides, produced if necessary. Prove that the rectangles CA . AM and CB . BL are together equal to the square of AB.

7. From an angular point A of the triangle ABC, the line AL is drawn to the middle point of the side BC. Prove that the square of BC and four times the square of AL are together double of the squares of BA and AC.

8. If P and Q be two points on a circle whose centre is O, and A and B two points on any line drawn through O such that AO is equal to BO, prove that the sum of the squares of AP and BP is equal to the sum of the squares of AQ and BQ.

9. Prove that the sum of the squares of the sides of a

parallelogram is equal to that of the squares of its two diagonals.

10. Prove that if any point within a parallelogram be joined to the four vertices, the sum of the areas of each pair of opposite triangles thus formed will be equal to half the area of the parallelogram.

11. If a triangle be formed by joining the middle point of one of the non-parallel sides of a trapezoid to the extremities of the opposite side, prove that the area of the triangle will be half the area of the trapezoid.

12. If a point P be taken either without or within a parallelogram ABCD of which BD is a diagonal, prove that the area of the triangle PBD will be equal to either the sum or difference of the areas of the triangles PAB and PBC.

13. Prove that of all rectangles with equal perimeters the area of the square is the greatest.

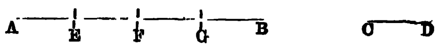
BOOK V.

SECTION I.—ON RATIO AND PROPORTION.

DEFINITION.

51.—When one magnitude is the sum of a number of parts each equal to another magnitude, the first magnitude is said to be a *multiple* of the second magnitude, and the second magnitude is said to be a *measure* or *aliquot part* of the first magnitude.

Fig. 1.



Thus if the straight line AB be made up of a number of parts AE, EF, FG, and GB, each equal to the straight line CD, then AB is said to be a multiple of CD, and CD to be an aliquot part of AB.

If, as in the illustration, the number of parts in AB were four, AB would be said to be four times CD, or CD multiplied by four, and conversely CD would be said to be one-fourth of AB, or AB divided by four.

In this case AB would be also said *to contain* CD four times, and CD *to be contained* in AB four times.

These relations are expressed as follows:

$$AB = 4 . CD \text{ or } 4CD$$
$$CD = \frac{1}{4} . AB \text{ or } \frac{AB}{4},$$

and similarly for any other numbers and magnitudes.

Note.—We apply the terms *sum* and *difference*, *multiple* and *measure*, to abstract numbers as well as to concrete

magnitudes, like lines, areas, and so forth. Thus, 5 is said to be the sum of 3 and 2, and 2 is said to be the difference of 5 and 3, by which we mean that if any magnitude were multiplied by 3 and by 2 the sum of the results would be the same as if the magnitude were multiplied by 5, and if any magnitude were multiplied by 5 and by 3 the difference of the results would be the same as if the magnitude were multiplied by 2.

So, also, the number 12 is said to be a multiple of the number 4, and the number 4 to be a measure of the number 12; by which we mean that if any magnitude be added together 12 times, the resulting sum is a multiple of the same magnitude added together 4 times, and we say that 12 is equal to 4 multiplied by 3, or to 3 times 4.

For the sake of brevity we frequently use letters for numbers; thus, a, b, c, &c., are frequently used as abbreviations for the phrase '*any number*,' and in such cases the result of multiplying the number b by the number a is represented by ab.

It is assumed as axiomatic that if any magnitude be multiplied by one number, and the product thus obtained be multiplied by a second number, the result will be the same as if the magnitude were first of all multiplied by the second number and this product were then multiplied by the first number, or that any magnitude multiplied by ab is equal to the same magnitude multiplied by ba, and therefore we say that ab is equal to ba.

It is also assumed that if A and B be any two magnitudes, and a any number, then $a\text{A}+a\text{B}$ will be equal to the sum of A and B multiplied by a, and that $a\text{A}\sim a\text{B}$ will be equal to the difference of A and B multiplied by a; or, in other words, that *if we add or subtract magnitudes and multiply the sum or difference thus obtained by any number, or if we multiply the same magnitudes by any number and then add or subtract the products, the result will be the same.*

DEFINITIONS.

52.—When each of two magnitudes is a multiple of, or exactly contains, a third magnitude, the first two magnitudes are said to be *commensurable*, and the third magnitude is called a *common measure* of the first two magnitudes.

53.—If there be four magnitudes such that the first contains the second as many times as the third contains the fourth, then the first and third magnitudes are said to be *equimultiples* of the second and fourth.

54.—The *ratio* of two commensurable magnitudes is the answer to the question how many times they respectively contain any common measure.

55.—In speaking of the ratio of two magnitudes, the magnitude first mentioned is called the *antecedent*, and that last mentioned the *consequent* of the ratio.

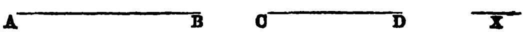
56.—The two numbers which express how many times the common measure is contained in the antecedent and consequent of a ratio respectively, are called the *numerator* and the *denominator* of the ratio.

A ratio is denoted :

1st. By writing the numerator above the denominator, or the antecedent above the consequent, with a line between them in each case.

2nd. By writing the numerator before the denominator, or the antecedent before the consequent, with the mark : between them in each case.

Fig. 2.



Thus, let AB and CD be two straight lines, and X a third straight line which is contained in AB and CD four times and three times respectively, then the ratio of AB to CD may have for numerator and denominator the numbers 4 and 3 respectively, and we should write :

$$\frac{AB}{CD} = \frac{4}{3} \qquad \frac{CD}{AB} = \frac{3}{4},$$

$$\text{or } AB : CD = 4 : 3, \qquad CD : AB = 3 : 4,$$

and similarly for any other numbers and magnitudes.

Note.—It is clear from the above definition that if the consequent of two magnitudes whose ratio is given be divided by the denominator of the ratio, and the part thus obtained be multiplied by the numerator, the resulting magnitude will be equal to the antecedent.

The term *multiply* is often used in an extended sense, so that, whether the antecedent be a *multiple* of the consequent or not, the antecedent is said to be equal to the consequent multiplied by the ratio between them, and the consequent is said to be equal to the antecedent divided by that ratio.

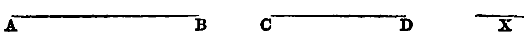
Thus in the preceding illustration, we should say that AB was equal to CD multiplied by $\frac{4}{3}$, and we should write as follows:

AB is equal to $\frac{4}{3}$ CD, and CD is equal to AB $\div \frac{4}{3}$.

PROPOSITION I.

The antecedent of any ratio multiplied by the denominator of that ratio is equal to the consequent multiplied by the numerator.

Fig. 3.



Let AB and CD be two straight lines such that a third straight line X is contained by them m times and n times respectively, and therefore such that

$$\frac{AB}{CD} \text{ is equal to } \frac{m}{n}.$$

Because AB is equal to mX

therefore nAB is equal to $n \times m$X, i.e. to nmX.

Similarly, mCD is equal to $m \times n$X, i.e. to mnX;

therefore nAB is equal to mCD,

and a similar proof is applicable in the case of any other commensurable magnitudes besides straight lines.

Corollary 1.—When the denominator of a ratio is *one*, the antecedent is equal to the consequent multiplied by the numerator, and, conversely, when the antecedent is a multiple of the consequent the ratio may have one for denominator, and the number of times the antecedent contains the consequent for numerator.

Thus, in the illustration to Definition 1 the ratio $\frac{AB}{CD}$ may be represented by $\frac{4}{1}$.

In such a case as this the denominator is generally omitted, and the numerator is said to be the ratio.

Corollary 2.—If the first of four magnitudes be the same multiple of the second that the third is of the fourth, the ratio of the first magnitude to the second will be the same as the ratio of the third magnitude to the fourth.

Corollary 3.—When the numerator of a ratio is equal to the denominator, the antecedent is equal to the consequent, for in this case a multiple of the antecedent is equal to the same multiple of the consequent. The ratio in this case is called *one* or unity.

PROPOSITION 2.

When a ratio is given and also a magnitude of any kind, then one, and only one, other magnitude of like kind can always be found so as to form with the given magnitude the antecedent and consequent of the given ratio.

Let $\frac{m}{n}$ be the given ratio, and let the straight line CD

be the given magnitude. Divide CD into n equal parts, and let X be equal to one of them.

Fig. 4.

Then one straight line AB, and no more than one straight line, can always be found equal to mX.

Therefore one straight line AB, and only one straight line, can always be found such that

$$\frac{\text{AB}}{\text{CD}} \text{ is equal to } \frac{m}{n}.$$

The given magnitude may be any other than a straight line without affecting the proof.

DEFINITIONS.

57.—If there be two ratios such that when equal magnitudes are taken as consequents in each ratio, the antecedents are also equal, then the ratios are said to be *equal* to one another; but if the corresponding antecedents be *unequal*, that ratio which has the greater antecedent is called the *greater* of the two ratios.

58.—If there be two ratios such that when equal magnitudes are taken as consequents in each ratio, the antecedent of one ratio is any multiple of the antecedent of the other, then the one ratio is said to be that same *multiple* of the other ratio.

Fig. 5.

Thus if AB, CD, and X be three straight lines such that AB is equal to m times CD, then the ratio $\frac{\text{AB}}{\text{X}}$ is said to be m times the ratio $\frac{\text{CD}}{\text{X}}$.

PROPOSITION 3.

If the denominators of two ratios be equal to one another, but the numerator of the second ratio be any multiple of the numerator of the first ratio, then the second ratio shall be that same multiple of the first ratio, and if the numerators of two ratios be equal to one another, but the denominator of the second ratio be any multiple of the denominator of the first ratio, then the first ratio shall be that same multiple of the second ratio.

1. Let there be two ratios, each having the number represented by y for denominator, and let the numerator of the first ratio be x, and that of the second ratio mx, then shall the ratio $\frac{mx}{y}$ be equal to m times the ratio $\frac{x}{y}$.

Fig. 6.

Take any straight line X as a common consequent of the two ratios, and let the straight lines AB and CD be the corresponding antecedents, so that the two ratios are $\frac{AB}{X}$ and $\frac{CD}{X}$ respectively.

Because x and y are the numerator and denominator respectively of the ratio $\frac{AB}{X}$,

And mx and y are the numerator and denominator respectively of the ratio $\frac{CD}{X}$;

therefore if X be divided into y equal parts, AB will contain x, and CD will contain mx of these parts;

therefore CD is equal to m times AB;

therefore $\frac{CD}{X}$ is equal to m times $\frac{AB}{X}$ (Def. 58),

i.e. $\frac{mx}{y}$ is equal to m times $\frac{x}{y}$.

2. Let each of the ratios have the number x for numerator, and let their denominators be y and my, then shall the ratio $\frac{x}{y}$ be m times the ratio $\frac{x}{my}$.

Fig. 7.

Take any straight line X as a common consequent of the two ratios, and let the straight lines AB and CD be the corresponding antecedents, so that the two ratios are $\frac{AB}{X}$ and $\frac{CD}{X}$ respectively.

As in case 1, if X be divided into y equal parts, AB will contain x of these parts; and if X be divided into my equal parts, CD will contain x of these parts.

But each of the y equal parts of X must be m times as large as each of the my equal parts of the same straight line;

therefore x of the y equal parts is m times as large as x of the my equal parts;

therefore AB is equal to m times CD;

therefore $\frac{AB}{X}$ is equal to m times $\frac{CD}{X}$ (Def. 58),

i.e. $\frac{x}{y}$ is equal to m times $\frac{x}{my}$.

Corollary.—When the numerator and denominator of a ratio are each multiplied or each divided by the same number, the ratio remains unaltered.

Note.—Since a ratio remains unaltered when any equimultiples of the numerator and denominator are written for those numbers respectively, it follows that the same ratio may be expressed in an infinite number of different ways.

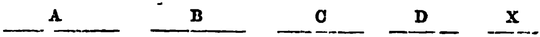
Thus, $\frac{3}{4}$, $\frac{6}{8}$, $\frac{9}{12}$, $\frac{12}{16}$, &c., all denote the same ratio.

Hence when we speak of *the* numerator and denominator of any given ratio, we mean *any one* of the pairs of numbers which might be selected, each denoting the given ratio.

PROPOSITION 4.

Magnitudes have to one another the same ratio which their equimultiples have.

Fig. 8.



Let there be four magnitudes as, for instance, the straight lines A, B, C, and D, such that A and B are equal to p times C and p times D respectively.

Let X be a straight line, such that C and D are equal to m times X and n times X respectively, and therefore such that $\frac{C}{D}$ is equal to $\frac{m}{n}$.

Because C is equal to mX, and D is equal to nX,

therefore A is equal to pmX, and B is equal to pnX;

therefore $\frac{A}{B}$ is equal to $\frac{pm}{pn}$ (Def. 54);

therefore $\frac{A}{B}$ is equal to $\frac{m}{n}$ (Prop. 3, *Cor.*),

i.e. $\frac{A}{B}$ is equal to $\frac{C}{D}$.

PROPOSITION 5.

Any two ratios may be made to have the same denominator or the same numerator without altering the value of either ratio.

Let the numerator and denominator of the first ratio be a and b, and those of the second ratio c and d respectively, so that the ratios are $\frac{a}{b}$ and $\frac{c}{d}$ respectively.

By the Corollary to Proposition 3, the ratio $\frac{a}{b}$ is not

altered by multiplying both the numerator and denominator by d,

$$\text{therefore } \frac{a}{b} \text{ is equal to } \frac{da}{db}.$$

Similarly $\frac{c}{d}$ is equal to $\frac{bc}{bd}$.

But db is equal to bd,

therefore $\frac{da}{db}$ and $\frac{bc}{bd}$ are respectively equal to $\frac{a}{b}$ and $\frac{c}{d}$, and

they have a common denominator db or bd.

Again, by the Corollary to Proposition 3,

$$\frac{a}{b} \text{ is equal to } \frac{ca}{cb},$$

$$\text{and } \frac{c}{d} \text{ is equal to } \frac{ac}{ad},$$

therefore $\frac{ca}{cb}$ and $\frac{ac}{ad}$ are respectively equal to $\frac{a}{b}$ and $\frac{c}{d}$,

and they have a common numerator ca or ac.

Corollary 1.—The proof may be extended to any number of ratios by multiplying every numerator in succession by all the denominators, except its own, for a new numerator, and multiplying all the denominators together for a common denominator, and similarly for a common numerator.

Corollary 2.—If two ratios be equal to one another they may, without altering their values, be made to have equal numerators and equal denominators.

DEFINITION.

59.—If the same magnitude be taken as consequent to each of two given ratios, the ratio of the sum of the antecedents to this common consequent is called *the sum* of the ratios, and the ratio of the difference of the antecedents to this common consequent is called *the difference* of the ratios.

The addition and subtraction of ratios are denoted respectively by the signs + and ~, or −, as in the case of concrete magnitudes.

Note.—Although the addition and subtraction of ratios is denoted by the same signs or marks as the addition and subtraction of concrete magnitudes such as lines, areas, angles, and so forth, yet it is very important to observe that there is a wide difference between the cases.

A concrete magnitude is divisible into parts, so that a larger magnitude is made up of the sum of smaller magnitudes of like kind, and hence it follows as axiomatic that if A and B be concrete magnitudes of like kind, then $A+B\sim B$ is equal to A, and if B be smaller than A, then $A\sim B+B$ is equal to A.

But a ratio is not divisible into parts; it is, as it has been defined to be, the answer to a question, and all the propositions connected with the addition and subtraction of ratios must be deduced rigorously from the definition (59). It must also be remembered that a ratio may in some cases be expressed by one number only (see Prop. 1, *Cor.* 1), and conversely that every number is a ratio, and therefore that the definitions of sum, difference, and multiple of ratios ought to include those of the sum, difference, and multiple of numbers. A reference to (Def. 51, *Note*) will easily show that this is the case.

PROPOSITION 6.

If two ratios be equal to two other ratios, each to each, then the sum or difference of the first two shall be equal respectively to the sum or difference of the last two; and if from the sum of two ratios one of them be subtracted, the remainder shall be equal to the other ratio.

1. If there be two ratios equal to two other ratios each to

Fig. 9.

each, then the sum or difference of the first two shall be equal respectively to the sum or difference of the last two.

Take any magnitude as a common consequent of all the ratios, as, for instance, the straight line X, and let the corresponding antecedents be the straight lines A, B, C, and D respectively, so that the ratios are $\frac{A}{X}$, $\frac{B}{X}$, $\frac{C}{X}$, and $\frac{D}{X}$.

Because $\frac{A}{X}$ is equal to $\frac{C}{X}$ and $\frac{B}{X}$ to $\frac{D}{X}$,

therefore A is equal to C and B to D (Def. 57),

therefore $A+B$ is equal to $C+D$ and $A \sim B$ is equal to $C \sim D$,

therefore $\frac{A+B}{X}$ is equal to $\frac{C+D}{X}$ and $\frac{A \sim B}{X}$ is equal to $\frac{C \sim D}{X}$,

therefore $\frac{A}{X}+\frac{B}{X}$ is equal to $\frac{C}{X}+\frac{D}{X}$,

and $\frac{A}{X} \sim \frac{B}{X}$ is equal to $\frac{C}{X} \sim \frac{D}{X}$.

2. If from the sum of two ratios one of them be subtracted the remainder shall be equal to the other ratio.

Fig. 10.

Take any magnitude, as for instance the straight line X, as a common consequent of both the ratios, and let the corresponding antecedents be the straight lines A and B, so that the ratios are $\frac{A}{X}$ and $\frac{B}{X}$ respectively.

Because $\frac{A}{X}+\frac{B}{X}$ is equal to $\frac{A+B}{X}$ (Def. 59),

therefore $\frac{A}{X}+\frac{B}{X} \sim \frac{B}{X}$ is equal to $\frac{A+B}{X} \sim \frac{B}{X}$.

But $\frac{A+B}{X} \sim \frac{B}{X}$ is equal to $\frac{A+B \sim B}{X}$ (Def. 59).

and $A+B \sim B$ is equal to A,

therefore $\frac{A+B}{X} \sim \frac{B}{X}$ is equal to $\frac{A}{X}$,

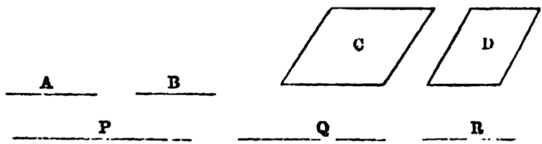
therefore $\frac{A}{X}+\frac{B}{X} \sim \frac{B}{X}$ is equal to $\frac{A}{X}$.

Corollary.—Similarly it may be proved that, if to the difference of two ratios the smaller be added, then the sum will be equal to the larger ratio.

DEFINITION.

60.—If there be two given ratios, and three commensurable magnitudes be taken, such that the ratio of the first of these magnitudes to the second is equal to the first of the given ratios, and the ratio of the second of these magnitudes to the third is equal to the second of the given ratios, then the ratio of the first of these magnitudes to the third is said to be the *product* of the second ratio multiplied by the first ratio, or to be the ratio *compounded* of the two given ratios.

Fig. 11.



Thus, let A and B be two straight lines, and C and D two areas, and let the three straight lines P, Q, and R be taken such that the ratio $\frac{P}{Q}$ is equal to the ratio $\frac{A}{B}$ and the ratio $\frac{Q}{R}$ to the ratio $\frac{C}{D}$, then the ratio $\frac{P}{R}$ is said to be the product of the ratio $\frac{C}{D}$ multiplied by the ratio $\frac{A}{B}$.

This product is denoted, as in the case of a magnitude multiplied by a number, by the sign $\times$ or . , thus, $\frac{A}{B} \times \frac{C}{D}$ or $\frac{A}{B} \cdot \frac{C}{D}$.

The product of two equal ratios is sometimes called the

square of either ratio, thus, $\frac{A}{B} \times \frac{A}{B}$ is called the square of $\frac{A}{B}$ and is written $\left(\frac{A}{B}\right)^2$

PROPOSITION 7.

If there be two given ratios, and a third ratio be formed having for its numerator the product of the numerators of the two given ratios, and for its denominator the product of their denominators, then this ratio shall be equal to the product of the two given ratios.

Fig. 12.

Let the numerator and denominator of the first ratio be represented by a and b, and those of the second ratio by c and d respectively, and take three commensurable magnitudes, as, for instance, the straight lines AB, CD, and EF, such that $\frac{AB}{CD}$ is equal to $\frac{a}{b}$ and $\frac{CD}{EF}$ is equal to $\frac{c}{d}$, then by (Def. 60) $\frac{AB}{EF}$ is equal to $\frac{a}{b} \times \frac{c}{d}$.

Because b . AB is equal to a . CD (Bk. V. Prop. 1),
and d . CD is equal to c . EF,
therefore db . AB is equal to da . CD,
and ad . CD is equal to ac . EF.
But da . CD is equal to ad . CD,
therefore db . AB is equal to ac . EF, i.e. to ca . EF,
therefore $\frac{AB}{EF}$ is equal to $\frac{ca}{db}$.

Corollary 1.—If A, B, C, and D be four commensurable magnitudes of like kind, $\frac{A}{B} \times \frac{C}{D}$ is equal to $\frac{A}{D} \times \frac{C}{B}$.

Corollary 2.—If any ratio be multiplied by the ratio of

two equal magnitudes to one another, i.e. by *one* (see Prop. 1, *Cor.* 3), the ratio remains unaltered.

Corollary 3.—The product of any one ratio multiplied by a second ratio is equal to the product of the second ratio multiplied by the first ratio.

Corollary 4.—If two ratios be equal to two other ratios, each to each, then the product of the first pair of ratios will be equal to the product of the second pair.

Similarly for any number of equal ratios.

DEFINITION.

61.—If there be three given ratios such that the product of the first multiplied by the second is equal to the third, then the first ratio is said to be the *quotient* of the third ratio *divided* by the second ratio.

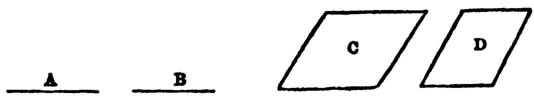
Thus if $\frac{C}{D} \times \frac{A}{B}$ be equal to $\frac{E}{F}$, then $\frac{A}{B}$ is said to be the quotient of $\frac{E}{F}$ divided by $\frac{C}{D}$, and this relation is written

$$\frac{A}{B} \text{ is equal to } \frac{E}{F} \div \frac{C}{D}.$$

PROPOSITION 8.

The quotient of one ratio divided by another is equal to the product of the one ratio multiplied by the ratio formed by interchanging the antecedent and consequent of the other ratio.

Fig. 13.



Let any pair of commensurable magnitudes, as, for instance, the straight lines A and B, be the antecedent and consequent of the one ratio, and let any other pair of commensurable

magnitudes, as, for instance, the areas C and D, be the antecedent and consequent of the other ratio.

Because $\frac{C}{D} \times \frac{D}{C}$ is equal to $\frac{C}{C}$ (Def. 60),

therefore $\frac{C}{D} \times \frac{D}{C}$ is equal to 1,

therefore $\frac{C}{D} \times \frac{D}{C} \times \frac{A}{B}$ is equal to $\frac{A}{B}$ (Prop. 7, *Cor.* 2), i. e. if the product of the ratios $\frac{A}{B}$ and $\frac{D}{C}$ be multiplied by the ratio $\frac{C}{D}$ the resulting ratio will be equal to $\frac{A}{B}$,

therefore by (Def. 61) the product of the ratio $\frac{A}{B}$ multiplied by the ratio $\frac{D}{C}$ is equal to the quotient of the ratio $\frac{A}{B}$ divided by the ratio $\frac{C}{D}$.

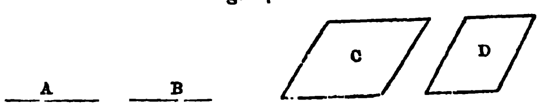
Corollary 1.—It has been proved in the course of the foregoing proposition that the product of any given ratio multiplied by the ratio formed by interchanging the numerator and denominator of the given ratio is equal to *one*.

Corollary 2.—If a and b be two numbers the quotient of the ratio a divided by the ratio b is the ratio $\frac{a}{b}$.

DEFINITION.

62.—When there are two pairs of commensurable magnitudes, such that the ratio between the first pair is equal to that between the second pair, the four magnitudes are said to be *proportionals* or *in proportion*.

Fig. 14.



Thus, if A and B be any two magnitudes of like kind, as,

for instance, two straight lines, and C and D be any two magnitudes of like kind, as, for instance, two areas, such that

$$\frac{A}{B} \text{ is equal to } \frac{C}{D},$$

then A, B, C, and D are said to be proportionals or in proportion.

This relation is also written thus:

$$A : B :: C : D.$$

All four magnitudes *may be* of the same kind, as, for instance, four straight lines, four areas, four angles, and so on, but at any rate the magnitudes in each pair taken separately, *must be* of the same kind.

When all four magnitudes are of the same kind, and moreover the second and third are equal to each other, the second or third magnitude is said to be a *mean proportional* between the first and fourth.

PROPOSITION 9.

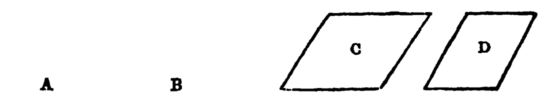
If the ratio of the first of four magnitudes to the second be equal to that of the third to the fourth, then:

1. *The ratio of the second to the first shall be equal to that of the fourth to the third;*

2. *The ratio of the sum or difference of the first and second to the second shall be respectively equal to the ratio of the sum or difference of the third and fourth to the fourth;*

3. *The ratio of the sum or difference of the first and third to the first shall be respectively equal to the ratio of the sum or difference of the second and fourth to the third.*

Fig. 15.



Let A and B be two commensurable magnitudes, as, for example, two straight lines, and let C and D be two com-

mensurable magnitudes, as, for example, two areas, and let the ratio $\frac{A}{B}$ be equal to the ratio $\frac{C}{D}$ then:

1. The ratio $\frac{B}{A}$ shall be equal to the ratio $\frac{D}{C}$.

Because $\frac{A}{B}$ is equal to $\frac{C}{D}$,

therefore $\frac{B}{A} \times \frac{A}{B} \times \frac{D}{C}$ is equal to $\frac{D}{C} \times \frac{C}{D} \times \frac{B}{A}$.

Also $\frac{B}{A} \times \frac{A}{B}$ and $\frac{D}{C} \times \frac{C}{D}$ are each equal to one,

therefore $\frac{D}{C}$ is equal to $\frac{B}{A}$ (Bk. V. Prop. 7, *Cor.* 2).

2. The ratio $\frac{A+B}{B}$ shall be equal to the ratio $\frac{C+D}{D}$.

Because $\frac{A}{B}$ is equal to $\frac{C}{D}$ and $\frac{B}{B}$ is equal to $\frac{D}{D}$ (each being *one*),

therefore $\frac{A}{B} + \frac{B}{B}$ is equal to $\frac{C}{D} + \frac{D}{D}$ (Bk. V. Prop. 6),

therefore $\frac{A+B}{B}$ is equal to $\frac{C+D}{D}$ (Def. 59).

Similarly, $\frac{A \sim B}{B}$ is equal to $\frac{C \sim D}{D}$.

3. The ratio $\frac{A+B}{A}$ shall be equal to the ratio $\frac{C+D}{C}$.

Because $\frac{A+B}{B}$ is equal to $\frac{C+D}{D}$,

and $\frac{B}{A}$ is equal to $\frac{D}{C}$ (Part I.),

therefore $\frac{A+B}{B} \times \frac{B}{A}$ is equal to $\frac{C+D}{D} \times \frac{D}{C}$

(Bk. V. Prop. 7, *Cor.* 4),

i. e. $\frac{A+B}{A}$ is equal to $\frac{C+D}{C}$

Similarly, $\frac{A \sim B}{A}$ is equal to $\frac{C \sim D}{C}$.

It is clear that A, B, C, and D may represent any other kind of magnitudes besides lines and areas, and the proof will remain unaffected.

PROPOSITION 10.

If four magnitudes of the same kind be proportionals, then shall the ratio of the first to the third be equal to the ratio of the second to the fourth.

Fig. 16.

Let A, B, C, and D be four magnitudes of the same kind, as, for instance, four straight lines, and such that the ratio $\frac{A}{B}$ is equal to the ratio $\frac{C}{D}$, then shall the ratio $\frac{A}{C}$ be equal to the ratio $\frac{B}{D}$.

Because $\frac{A}{B}$ is equal to $\frac{C}{D}$,

therefore $\frac{B}{C} \times \frac{A}{B}$ is equal to $\frac{B}{C} \times \frac{C}{D}$.

But $\frac{B}{C} \times \frac{A}{B}$ is equal to $\frac{A}{B} \times \frac{B}{C}$, i.e. to $\frac{A}{C}$ (Bk. V. Prop. 7, *Cor.* 2 and Def. 60).

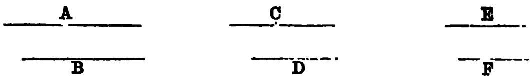
And $\frac{B}{C} \times \frac{C}{D}$ is equal to $\frac{B}{D}$,

therefore $\frac{A}{C}$ is equal to $\frac{B}{D}$.

PROPOSITION 11.

If there be any number of equal ratios in which all the antecedents and consequents are of the same kind, then shall the ratio of the sum of the antecedents to the sum of the consequents be equal to any one of the equal ratios.

Fig. 17.



Let A, C, and E be any three magnitudes of the same kind, as, for example, three straight lines, and let B, D, and F be three other magnitudes of the same kind as A, C, and E, and such that the ratios $\frac{A}{B}$, $\frac{C}{D}$, and $\frac{E}{F}$ are all equal, then shall the ratio $\frac{A+C+E}{B+D+F}$ be equal to any one of the ratios $\frac{A}{B}$, $\frac{C}{D}$, and $\frac{E}{F}$.

Because the ratios $\frac{A}{B}$, $\frac{C}{D}$, and $\frac{E}{F}$ are all equal,

therefore they may be expressed with the same numerator and the same denominator without altering their values (Bk. V. Prop. 5, *Cor.* 2).

Let this common numerator and denominator be m and n respectively.

Then $n \,.\, A$ is equal to $m \,.\, B$.

$n \,.\, C$ „ $m \,.\, D$.

$n \,.\, E$ „ $m \,.\, F$.

therefore $n \,.\, A + n \,.\, C + n \,.\, E$ is equal to $m \,.\, B + m \,.\, D + m \,.\, F$.

therefore n times the sum $A+C+E$ is equal to m times the sum $B+D+F$,

therefore $\frac{A+C+E}{B+D+F}$ is equal to $\frac{m}{n}$,

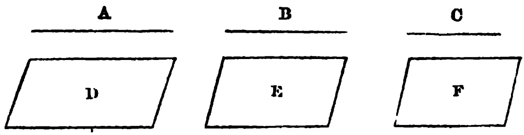
i.e. $\frac{A+C+E}{B+D+F}$ is equal to $\frac{A}{B}$ or $\frac{C}{D}$ or $\frac{E}{F}$;

and the same is true whatever be the number of equal ratios and whatever be the magnitudes considered.

PROPOSITION 12.

If there be any number of magnitudes, and as many others, which taken two and two in order have the same ratio, the first shall have to the last of the first set of magnitudes the same ratio as the first has to the last of the second set of magnitudes.

Fig. 18.



Let A, B, and C be three magnitudes of the same kind, as, for instance, three straight lines, and let D, E, and F be three magnitudes of the same kind, as, for instance, three areas, and let the ratio $\frac{A}{B}$ be equal to the ratio $\frac{D}{E}$, and the ratio $\frac{B}{C}$ be equal to the ratio $\frac{E}{F}$, then shall the ratio $\frac{A}{C}$ be equal to the ratio $\frac{D}{F}$.

Because $\frac{A}{B}$ is equal to $\frac{D}{E}$,

and $\frac{B}{C}$ is equal to $\frac{E}{F}$,

therefore $\frac{A}{B} \times \frac{B}{C}$ is equal to $\frac{D}{E} \times \frac{E}{F}$ (Bk V. Prop. 7, *Cor.* 4).

But $\frac{A}{B} \times \frac{B}{C}$ is equal to $\frac{A}{C}$ (Def. 61),

also $\frac{D}{E} \times \frac{E}{F}$ is equal to $\frac{D}{F}$,

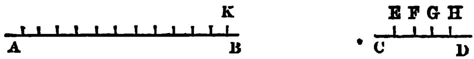
therefore $\frac{A}{C}$ is equal to $\frac{D}{F}$.

The same proof is clearly applicable whatever be the number and kind of magnitudes in each set.

Note.—I. It may happen (see Sect. II.) that two magnitudes of the same kind have no common measure; in such a case they are said to be *incommensurable.*

The definition of ratio given above does not apply directly to two incommensurable magnitudes, but it may be extended to this case.

Fig. 19.



Let there be two incommensurable magnitudes, as, for instance, the two straight lines AB and CD.

Divide one of these lines, as CD, into any number of equal parts, CE, EF, FG, GH, and HD, and set off upon AB, beginning from A, lines each equal to one of these equal parts of CD until a point K is reached, such that KB is less than one of these parts.

Because AK and CD are each multiples of the same straight line CE, therefore AK and CD are commensurable.

Increase the number of parts into which CD is divided, then the length of each of these parts will be correspondingly *diminished*, and the last point of division of AB or K, will approach continually nearer and nearer to the extremity B, because the distance KB is always less than each of the parts of CD.

As there is no limit to the number of parts into which CD may be divided, it follows that we may always set off upon AB a line commensurable with CD, and differing from AB by as small a line as we please.

Hence in the case of two incommensurable straight lines we may always find a straight line differing from one of them by a length less than any assignable length, and at

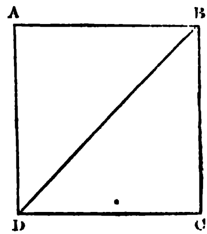
the same time commensurable with the other line, or we may always find two straight lines differing respectively from the two incommensurable straight lines by lengths less than any assignable length, and at the same time commensurable with one another.

The *limiting value* of the ratio of these two commensurable straight lines is called the ratio of the two incommensurable straight lines.

And similarly for any other incommensurable magnitudes.

It follows from what has been said that no *finite* numerator and denominator however large can express the ratio of two incommensurable magnitudes, but that by *properly* increasing both numerator and denominator we may obtain a ratio as nearly equal as we please to the required ratio, i.e. such that the antecedent and consequent of the ratio thus obtained may differ respectively from the given incommensurable magnitudes by as small a quantity as we please.

Fig. 20.



If ABCD be a square, and BD one of the diagonals, then BD and AB are, as will be proved in the next section, incommensurable magnitudes.

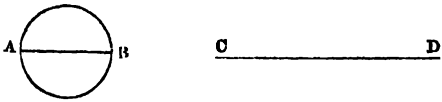
The ratios whose numerators and denominators are 14 and 10, 141 and 100, 1414 and 1000, 14142 and 10000, and so on, approach continually nearer and nearer to the value of the ratio of BD to AB.

That is to say, if AB were divided into ten equal parts, BD would contain fourteen such parts and a remainder over less than one of these tenth parts of AB; or, in other words, if a line were taken having to AB the ratio $\frac{14}{10}$, this line would differ from BD by less than the tenth part of AB.

Similarly, if a line were taken having to AB the ratio $\frac{141}{100}$, this line would differ from BD by less than the hundredth part of AB, and so on.

Again, if CD be the length of the circumference of the circle of which AB is the diameter (see *Note* at end of

Fig. 21.



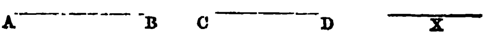
Book II.), it may be proved that CD and AB are incommensurable magnitudes, and that the ratios $\frac{31}{10}$, $\frac{314}{100}$, $\frac{3141}{1000}$, &c., are successive approximations to the ratio $\frac{CD}{AB}$.

That is to say, if a line were taken having to AB the ratio $\frac{31}{10}$, this line would differ from CD by less than the tenth part of AB; and if a line were taken having to AB the ratio $\frac{3141}{1000}$, this line would differ from CD by less than the thousandth part of AB.

In geometry we are as often concerned with the investigation of the relations of incommensurable as of commensurable magnitudes.

II. When there are any number of magnitudes, and each magnitude is expressed as the product of one of them multiplied by some known ratio (*Note*, p. 146 above), that magnitude in terms of which each of them is expressed is called the *unit* of magnitude of the particular kind under consideration, and the different ratios are called the *numerical values* of the different magnitudes.

Fig. 22.



Thus, let AB, CD, &c., be the lengths of certain lines, and let X be another line, as a foot or an inch, then if the ratio of AB to X were 5, that of CD to X $\frac{16}{7}$, and so on, the *numerical values* of the lines AB, CD, &c., would be said

to be 5, $\frac{15}{7}$, and so on, and the line X would be said to be the unit of length.

III. We have seen (*Note* to Def. 54) that the terms *multiple, measure, equimultiple,* &c., are applied to the particular kinds of ratios which can be denoted by single numbers, and it is important to remark that these terms are extended to all kinds of ratios without restriction.

Thus, $\frac{7}{4}$ is said to be a *multiple* of $\frac{7}{8}$, because it is equal to $2 \times \frac{7}{8}$ (Bk. V. Prop. 3), and $\frac{5}{3}$ and $\frac{9}{7}$ are said to be *equimultiples* of $\frac{5}{6}$ and $\frac{9}{14}$ respectively.

So, also, the ratio of two commensurable ratios is defined to be the answer to the question how many times they respectively contain the same common measure.

Thus the ratio of $\frac{7}{2}$ and $\frac{35}{6}$ is $\frac{3}{5}$, because $\frac{7}{2}$ is equal to $3 \times \frac{7}{6}$, and $\frac{35}{6}$ is equal to $5 \times \frac{7}{6}$.

It is easily seen that the ratio of two ratios must be the same as the quotient of the antecedent divided by the consequent.

For if a and b be the numerator and denominator of the antecedent, and c and d those of the consequent, we know that the ratios are identical with the ratios $\frac{da}{db}$ and $\frac{bc}{bd}$ respectively (Bk. V. Prop. 5);

therefore they contain the ratio $\frac{1}{bd}$, ad and bc times respectively,

therefore their ratio is $\frac{ad}{bc}$ or $\frac{a}{b} \div \frac{c}{d}$ (Bk. V. Prop. 8).

The student may omit Section II. of this Book, and pass on immediately to Book VI.

Section II.—On the Commensurability and Incommensurability of Magnitudes of Like Kinds.

PROPOSITION 13.

If a magnitude be a measure of each of two other magnitudes, it shall be a measure both of the sum and difference of any multiples of these other magnitudes.

Fig. 23.

Let A, B, and C be any three magnitudes of like kind, as, for instance, three straight lines, and let C be a common measure of A and B, then C shall be also a measure of the sum or difference of any multiples of A and B.

Fig. 24.

Let EF be any multiple of A, and FG any multiple of B, and let EF and FG be placed in the same straight line with a common extremity at F; and first let FG be placed in the prolongation of EF, then the length of EG will be the sum of some multiple of A together with some multiple of B.

Let EF be divided into portions EH, HK, &c., each equal to A, in the points H, K, &c., then the last of these points will coincide with F, because EF is a multiple of A.

Similarly, let FG be divided into portions FL, LM, &c., each equal to B in the points L, M, &c., then the last of these points will coincide with G, because FG is a multiple of B.

If now portions be marked off upon EG equal to C, the points H, K, &c., will coincide with points of division, because EH, HK, &c., being equal to A are multiples of C, and therefore one point of division will coincide with F, and similarly the last point of division will coincide with G.

Therefore EG is a multiple of C, or C is a measure of EG.

Next, let FG lie upon EF, then EG will be the difference of some multiple of A and of some multiple of B.

If portions be marked off upon EG equal to C beginning with E, it follows from reasoning similar to the above that one point of division will fall upon E and another upon G;

therefore EG is a multiple of C, or C is a measure of EG;

therefore C is a measure both of the sum and difference of any multiples of A and B.

Similar reasoning applies to other like magnitudes besides the lengths of straight lines.

Note.—The following shorter proof may also be given of this proposition.

Let A be equal to C multiplied by the number represented by m, or let A be equal to mC, and let B be equal to nC.

Let p and q be any other numbers, and therefore $p\,.\,A$ and $q\,.\,B$ any multiples of A and B;

therefore $p\,.\,A$ is equal to $p \times m$ times C,

and $q\,.\,B$ is equal to $q \times n$ times C.

But $p \times mC + q \times nC$ is equal to C multiplied by the sum of the numbers pm and qn;

therefore $pA + qB$ is some multiple of C.

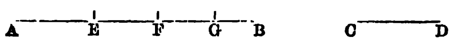
Similarly, also, $pA \sim qB$ is some multiple of C.

DEFINITION.

63.—When as many parts as possible, each equal to some smaller magnitude, are taken from a larger magnitude, the larger magnitude is said to be *divided* by the smaller, the number of parts taken from the larger is called the *quotient*, and the part of the larger which remains after the subtraction is called the *remainder* of the division of the larger by the smaller.

Also the smaller magnitude is called the *divisor*, and the larger the *dividend*.

Fig. 25.



Thus, if parts AE, EF, &c., each equal to the smaller straight line CD, be marked off upon the larger straight line AB until a portion GB remains less than CD, then AB is said to be divided by CD, the number of parts AE, EF, &c., equal to CD, is called the *quotient* of the division, and the portion GB is called the *remainder*.

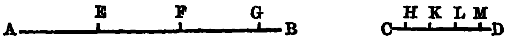
When GB is equal to zero, or the last point of division coincides with B, AB is a multiple of CD.

PROPOSITION 14.

If the larger of two like magnitudes be divided by the smaller, and then the divisor be divided by the remainder, and the process be continually repeated, the divisor and remainder at any step being the dividend and divisor of the next succeeding step, every common measure of the first dividend and first divisor shall be a common measure of each succeeding dividend and divisor, and conversely every common measure of any dividend and divisor shall be a common measure of the first dividend and first divisor.

For distinctness sake, let the two magnitudes be represented by two straight lines as AB and CD.

Fig. 26.



Let AB be divided by CD, and let the remainder be GB.

Let CD be divided by GB, and let the remainder be MD, and so on.

Then AG is a multiple of CD.

Because AG is a multiple of CD, therefore every common measure of AB and CD is a measure of AB $\sim$ AG, that is, of GB, and it is also a measure of CD;

therefore every common measure of AB and CD is a common measure of GB and CD.

Also every common measure of GB and CD is a common measure of the sum of GB and any multiple of CD.

But AG is a multiple of CD;

therefore every common measure of GB and CD is a common measure of AG+GB, that is, of AB, and it is also a measure of CD;

therefore every common measure of GB and CD is a common measure of AB and CD.

Similarly, it may be proved that every common measure of GB and CD, i.e. of AB and CD, is a common measure of MD and GB, and that every common measure of MD and GB is a common measure of GB and CD, i.e. of AB and CD.

And so on for any number of steps in the process.

PROPOSITION 15.

If at any step of the process described in the last Proposition the remainder becomes zero, the corresponding divisor shall be the greatest common measure of the two given magnitudes;

and if the remainder never becomes zero, however far the process may be continued, the two given magnitudes shall be incommensurable.

First, if at any step of the process the remainder becomes zero, the corresponding divisor must be a measure of the corresponding dividend;

therefore it must be a common measure of itself and that dividend;

therefore it is a common measure of the two given magnitudes (last Proposition).

But every common measure of the two given magnitudes is a common measure of this divisor and dividend;
therefore the greatest common measure of the two given magnitudes is the greatest common measure of this divisor and dividend.

But if the divisor be a common measure of itself and the dividend, it must be the greatest common measure of itself and the dividend;

therefore it will be the greatest common measure of the two given magnitudes.

Next, suppose that no divisor is ever a measure of any dividend, then the two magnitudes are incommensurable.

For (referring to the last Proposition) GB is less than CD, and AG is not less than CD; therefore GB is less than half AB;

therefore the first remainder is less than half the first dividend.

Similarly, any other remainder is less than half the corresponding dividend.

Therefore the remainders diminish without limit as the process is continually repeated.

But we have seen that no common measure of the two given magnitudes can be greater than the remainder at any step of the process;

therefore, if the process never terminates, no common measure can be found, however small;
therefore the given magnitudes are incommensurable.

PROPOSITION 16.

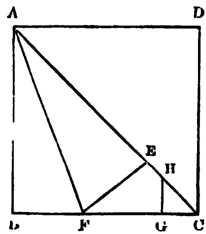
The diagonal and side of a square are incommensurable magnitudes.

Let ABCD be a square, and AC its diagonal.

Because ABC is a right angle, therefore AC is greater than AB.

Because two sides of a triangle are greater than the third side, therefore AC is less than AB+BC or less than 2AB.

Fig. 27.



Cut off AE upon AC equal to AB, then EC will be less than AB or BC;
therefore in the first step of the process the quotient is one and the remainder EC.

From E draw EF at right angles to AC, meeting BC in F, and join AF.

Because in the two right-angled triangles AEF and ABF, the hypothenuse AF is common, and the side AE equal to AB;

therefore EF is equal to BF.

Because the side BA is equal to the side BC, therefore the angle BAC is equal to the angle BCA.

Because ABC is a right angle, therefore the angles BAC and BCA are together equal to a right angle, and therefore since they are equal to one another, each of them is half a right angle.

Because ECF is half a right angle, and FEC a right angle, therefore EFC is half a right angle;

therefore EFC is equal to ECF, and the side EF to the side EC.

But EF is equal to BF; therefore EC is equal to BF.

If now we divide BC by EC, this being the second step in the process, the first division reckoning from B will be F, and the second will be G, such that FG is equal to FE, or the quotient will be 2, and the remainder GC.

But FC is clearly the diagonal of a square of which FE and EC are the two sides;

therefore the process is henceforward the same as the last, and thus each step continually repeats itself, and therefore we never arrive at a divisor which measures its corresponding dividend;

therefore AB and AC are incommensurable.

BOOK VI.

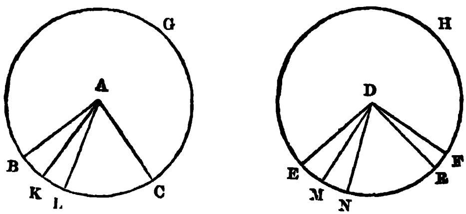
APPLICATION OF PROPORTION TO GEOMETRY.

SECTION I.—MISCELLANEOUS PROPOSITIONS.

PROPOSITION I.

In equal circles angles at the centres are to one another in the same ratio as the circumferences by which they are subtended.

Fig. 1.



Let BCG and EFH be equal circles whose centres are A and D.

Let BAC and EDF be any two angles at the centres A and D, and subtended by the arcs BC and EF respectively,

then shall the ratio $\frac{BAC}{EDF}$ be equal to the ratio $\frac{BC}{EF}$.

First, let the arcs BC and EF be *commensurable.*

Take BK, any common measure of BC and EF, and let EM be equal to BK.

Because BK is a measure of BC, therefore BC may be

divided into a certain number of equal arcs, BK, KL, &c., in the points K, L, &c.; and similarly, EF may be divided into a certain number of equal arcs, EM, MN, &c., in the points M, N, &c.

Join AK, AL, &c., DM, DN, &c.

Because the arcs BK, KL, &c. are all equal, therefore the angles BAK, KAL, &c., are all equal (Bk. II. Prop. 9), therefore the arc BC is the same multiple of the arc BK that the angle BAC is of the angle BAK;

therefore $\frac{BAC}{BAK}$ is equal to $\frac{BC}{BK}$ (Bk. V. Prop. 1, *Cor.* 2).

Similarly, $\frac{EDF}{EDM}$ is equal to $\frac{EF}{EM}$;

therefore $\frac{EDM}{EDF}$ is equal to $\frac{EM}{EF}$.

But BK is equal to EM, and consequently BAK is equal to EDM (Bk. II. Prop. 9);

therefore $\frac{BAK}{EDF}$ is equal to $\frac{BK}{EF}$,

and $\frac{BAC}{BAK}$ is equal to $\frac{BC}{BK}$,

therefore $\frac{BAC}{EDF}$ is equal to $\frac{BC}{EF}$ (Bk. V. Prop. 12).

Next, let BC and EF be *incommensurable.*

In this case we know from Book V. (*Note*, p. 164) that we may always find an arc ER *as nearly equal as we please to* EF, and such that BC and ER are commensurable.

Join DR, then, by the first part of this proposition, we know that

$$\frac{BAC}{EDR} \text{ is equal to } \frac{BC}{ER}.$$

And this being *always* true will be true in the limit when R moves up to and *as nearly as we please* coincides with F;

therefore $\frac{BAC}{EDF}$ is equal to $\frac{BC}{EF}$.

This proposition is true for angles of any magnitude

whatever, i.e. whether greater or less than two right angles (Bk. II. *Note*, p. 74).

Corollary 1.—Since the angle at the centre of a circle is double of the angle at the circumference subtended by the same arc, and since magnitudes have to one another the same ratio that their equimultiples have, it follows that in equal circles angles at the circumferences are to one another in the same ratio as the arcs by which they are subtended.

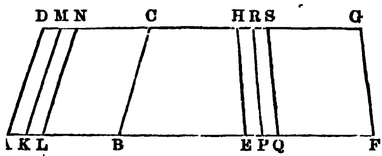
Corollary 2.—The same reasoning in all respects applies to angles and arcs in the *same* circle, whether the angles be at the centre or the circumference of that circle.

Corollary 3.—It follows from the reasoning of this proposition, that if there be four magnitudes such that, when the first and third are divided into the same number of equal parts, the *second* magnitude is the same multiple of each aliquot part of the *first*, which the *fourth* is of each aliquot part of the *third*, then the ratio of the first magnitude to the second is equal to the ratio of the third magnitude to the fourth.

PROPOSITION 2.

The areas of parallelograms and triangles between the same parallels are to one another in the same ratio as their bases.

Fig. 2.



1st. Let ABCD and EFGH be two parallelograms between the same parallels, AF and DG, then shall

$$\frac{\text{area of AC}}{\text{area of EG}} \text{ be equal to } \frac{\text{AB}}{\text{EF}}.$$

Let AB and EF be *commensurable.*

Take AK, any common measure of AB and EF, and let EP be equal to AK.

Because AK is a measure of AB therefore AB may be divided into a certain number of equal parts, AK, KL, &c., in the points K, L, &c.; and similarly, EF may be divided into a certain number of equal parts, EP, PQ, &c., in the points P, Q, &c.

Complete the parallelograms AM, KN, &c., ER, PS, &c., as in the figure.

Because the parallelograms AM, KN, &c., are on equal bases, and between the same parallels, therefore their areas are all equal (Bk. IV. Prop. 5),

therefore AB and the area of AC are equimultiples of AK and the area of AM.

Similarly, EF and the area of EG are equimultiples of EP and the area of ER.

But EP is equal to AK, and consequently the area of ER is equal to the area of AM;

therefore EF and the area of EG are equimultiples of the same aliquot parts of AB and the area of AC respectively;

$$\text{therefore } \frac{\text{area of AC}}{\text{area of EG}} \text{ is equal to } \frac{\text{AB}}{\text{EF}}$$

(Bk. VI. Prop. 1, *Cor.* 3).

The proof may be extended to the case in which AB and EF are incommensurable, by reasoning similar to that employed in the last proposition.

2nd. Because the area of a triangle is half the area of a parallelogram upon the same base and between the same parallels, and because magnitudes have the same ratio to one another which their equimultiples have, therefore the areas of triangles between the same parallels are to one another as their bases.

Corollary 1.—If ABCD and EFGH be two parallelograms having the angle at A equal to the angle at E, and the side AD equal to the side EH, and if the point K be taken upon

AB or AB produced so that AK is equal to EF, and if KL be parallel to AD, then the area of AL is equal to the area of EG (Bk. IV. Prop. 2);

Fig. 3.

$$\text{therefore } \frac{\text{area of AC}}{\text{area of EG}} \text{ is equal to } \frac{\text{area of AC}}{\text{area of AL}};$$

$$\text{therefore } \frac{\text{area of AC}}{\text{area of EG}} \text{ is equal to } \frac{\text{AB}}{\text{AK}} \text{ i.e. to } \frac{\text{AB}}{\text{EF}};$$

therefore *if two equiangular parallelograms have a side of the one equal to a side of the other, their areas shall be to another in the ratio of the remaining side of the one to the remaining side of the other.*

Corollary 2.—Since the area of any given parallelogram is equal to the area of the rectangular parallelogram whose adjacent sides are the base and altitude of the given parallelogram (Bk. IV. Prop. 9), it follows from *Cor.* 1 that the areas of two parallelograms with equal altitudes are to one another in the ratio of their bases, and the areas of two parallelograms with equal bases are to one another in the ratio of their altitudes, and similarly for the areas of two triangles.

PROPOSITION 3.

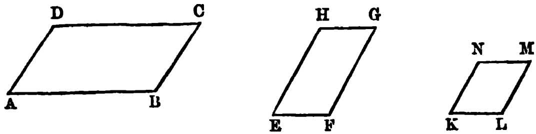
Equiangular parallelograms are to one another in the ratio compounded of the ratio of their sides.

Let ABCD and EFGH be two equiangular parallelograms, then shall

$$\frac{\text{the area of AC}}{\text{the area of EG}} \text{ be equal to } \frac{\text{AB}}{\text{EF}} \times \frac{\text{AD}}{\text{EH}}.$$

Let KLMN be a parallelogram equiangular with AC and EG, and having the sides KL and KN equal to EF and AD respectively.

Fig. 4.



By the last proposition, *Cor.* 1,

$$\frac{\text{area of AC}}{\text{area of KM}} \text{ is equal to } \frac{\text{AB}}{\text{KL}} \text{ i.e. to } \frac{\text{AB}}{\text{EF}},$$

$$\text{and } \frac{\text{area of KM}}{\text{area of EG}} \text{ is equal to } \frac{\text{KN}}{\text{EH}} \text{ i.e. to } \frac{\text{AD}}{\text{EH}};$$

$$\text{therefore } \frac{\text{area of AC}}{\text{area of EG}} \text{ is equal to } \frac{\text{AB}}{\text{EF}} \times \frac{\text{AD}}{\text{EH}}.$$

Corollary.—If the parallelograms become squares, AB is equal to AD and EF to EH, and therefore $\frac{\text{AB}}{\text{EF}} \times \frac{\text{AD}}{\text{EH}}$ is equal to $\frac{\text{AB}}{\text{EF}} \times \frac{\text{AB}}{\text{EF}}$;

therefore the squares upon two straight lines are to one another in the square of the ratio of the two lines (see Def. 60).

PROPOSITION 4.

If four straight lines be proportionals then the rectangle of the extremes shall be equal to the rectangle of the means; and if the rectangle of two straight lines be equal to the rectangle of two other straight lines, then these four straight lines shall be proportional, the adjacent sides of one of the rectangles being the extremes, and those of the other rectangle the means in the proportion.

1. Let A, B, C and D be four straight lines, such that the ratio $\frac{A}{B}$ is equal to the ratio $\frac{C}{D}$, then shall the rectangle of A and D be equal to the rectangle of B and C.

Fig. 5.

Because the rectangle of A and D is the area of the rectangular parallelogram of which A and D are adjacent sides, and similarly for the rectangle of B and C,

$$\text{therefore } \frac{\text{rectangle of A and D}}{\text{rectangle of B and C}} \text{ is equal to } \frac{A}{B} \times \frac{D}{C}$$

(last Prop).

$$\text{Because } \frac{A}{B} \text{ is equal to } \frac{C}{D} \text{ therefore } \frac{D}{C} \text{ is equal to } \frac{B}{A}$$

$$\text{therefore } \frac{\text{rectangle of A and D}}{\text{rectangle of B and C}} \text{ is equal to } \frac{A}{B} \times \frac{B}{A}.$$

But $\frac{A}{B} \times \frac{B}{A}$ is equal to 1 (Bk. V. Prop. 8, *Cor.* 1),

therefore the rectangle of A and D is equal to the rectangle of B and C.

2. Let A, B, C, and D be four straight lines, such that the rectangle of A and D is equal to the rectangle of B and C, then shall the ratio $\frac{A}{B}$ be equal to the ratio $\frac{C}{D}$.

If $\frac{A}{B}$ be not equal to $\frac{C}{D}$, let E be a straight line such that

$\frac{A}{B}$ is equal to $\frac{C}{E}$,

therefore the rectangle of A and E is equal to the rectangle of B and C, by the first part.

But by hypothesis the rectangle of A and D is equal to the rectangle of B and C,

therefore the rectangle of A and D is equal to the rectangle of A and E.

But if two parallelograms with equal areas be on the same base they must be also between the same parallels,

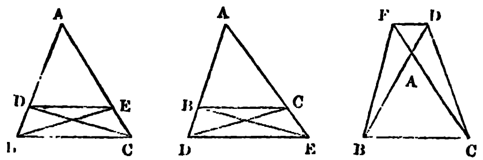
therefore D is equal to E,

therefore $\frac{A}{B}$ is equal to $\frac{C}{D}$.

PROPOSITION 5.

If a straight line be drawn parallel to one of the sides of a triangle it shall cut the other two sides or those sides produced proportionally; and if two of the sides, or the sides produced, of a triangle be cut proportionally, the straight line which joins the points of section shall be parallel to the remaining side of the triangle.

Fig. 6.



1st. Let the straight line DE be parallel to BC, one of the sides of the triangle ABC, then DE shall divide the sides AB and AC, or these sides produced proportionally, i.e. so that

$$\frac{AD}{DB} \text{ is equal to } \frac{AE}{EC}.$$

Join CD and BE.

Because the triangles BDE and CED are upon the same base DE, and between the same parallels DE and BC, therefore their areas are equal;

$$\text{therefore } \frac{\text{area of ADE}}{\text{area of BDE}} \text{ is equal to } \frac{\text{area of ADE}}{\text{area of CED}}.$$

But because ADE and BDE have the same altitude, viz. the perpendicular from E upon AB,

$$\text{therefore } \frac{\text{area of ADE}}{\text{area of BDE}} \text{ is equal to } \frac{\text{AD}}{\text{DB}}$$

(Bk. VI. Prop. 1, *Cor.* 2).

$$\text{Similarly, } \frac{\text{area of ADE}}{\text{area of CED}} \text{ is equal to } \frac{\text{AE}}{\text{EC}},$$

$$\text{therefore } \frac{\text{AD}}{\text{DB}} \text{ is equal to } \frac{\text{AE}}{\text{EC}},$$

$$\text{2nd. Let } \frac{\text{AD}}{\text{DB}} \text{ be equal to } \frac{\text{AE}}{\text{EC}},$$

then shall DE be parallel to BC.

Join BE and CD.

It may be proved, as in the first part, that

$$\frac{\text{area of ADE}}{\text{area of BDE}} \text{ is equal to } \frac{\text{AD}}{\text{DB}} \text{ and that}$$

$$\frac{\text{area of ADE}}{\text{area of CED}} \text{ is equal to } \frac{\text{AE}}{\text{EC}}.$$

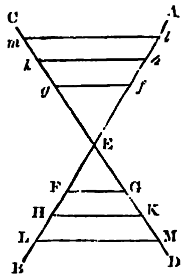
$$\text{But by hypothesis } \frac{\text{AD}}{\text{DB}} \text{ is equal to } \frac{\text{AE}}{\text{EC}},$$

$$\text{therefore } \frac{\text{area of ADE}}{\text{area of BDE}} \text{ is equal to } \frac{\text{area of ADE}}{\text{area of CED}},$$

therefore the area of BDE is equal to the area of CED, therefore the triangles BDE and CED are upon the same base and have the same area,

therefore DE is parallel to BC (Bk. IV. Prop. 8).

Fig. 7.



Corollary. — It follows from this Proposition that if two intersecting straight lines, as AB and CD, be cut by a series of parallel straight lines, as, for instance, FG, HK, LM, &c., on one side of their point of intersection, and by *fg*, *hk*, *lm*, &c., on the other side of that point, then they shall be divided proportionally,

that is to say, the ratio of any pair of segments of AB shall be equal to the ratio of the corresponding pair of segments of CD; so that

$$\frac{FH}{HL} \text{ is equal to } \frac{GK}{KM},$$

$$\frac{EH}{Ef} \text{ is equal to } \frac{EK}{Eg},$$

$$\frac{Fh}{Hl} \text{ is equal to } \frac{Gk}{Km}, \text{ and so on.}$$

For by the Proposition

$$\frac{FH}{EF} \text{ is equal to } \frac{GK}{EG},$$

$$\text{therefore } \frac{FH}{EH} \text{ is equal to } \frac{GK}{EK} \text{ (Bk. V. Prop. 9).}$$

$$\text{Also } \frac{LH}{EH} \text{ is equal to } \frac{MK}{EK},$$

$$\text{and } \frac{EH}{FH} \text{ is equal to } \frac{EK}{GK},$$

$$\text{therefore } \frac{LH}{FH} \text{ is equal to } \frac{MK}{GK},$$

and similarly for any other corresponding pairs of segments whatever.

Note.—When, as in this case, there are two sets of magnitudes, A, B, C, &c., and a, b, c, &c., such that

$$\frac{A}{B} \text{ is equal to } \frac{a}{b}, \quad \frac{B}{C} \text{ is equal to } \frac{b}{c}, \quad \frac{C}{D} \text{ is equal to } \frac{c}{d},$$

and so on, it is usual to express this relation as follows:

$$A : B : C, \text{ \&c. } :: a : b : c, \text{ \&c.}$$

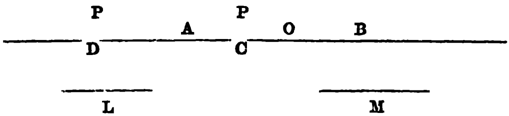
PROPOSITION 6.

If a given finite straight line be produced indefinitely in both directions, it shall be always possible to find two points, one situated upon the given straight line, and the other upon that line produced, such that the distances of either of these points from the extremities of the given straight line are to each

other in a certain given ratio; and both of these points shall be situated upon the same side of the point of bisection of the given line.

Let AB be a given finite straight line of which O is the middle point, and let L and M be two straight lines whose

Fig. 8.



ratio is equal to any given ratio, then if the straight line AB be produced indefinitely in both directions it shall be always possible to find two points, one on AB, and the other on AB produced, such that the ratio of the distances of each of these points from A and B respectively is equal to the ratio of L to M, and both of these points shall be situated on the same side of O.

Let L be less than M.

Because L is less than M, therefore L+M is greater than 2L,

therefore the ratio $\dfrac{L}{L+M}$ is less than $\frac{1}{2}$.

Suppose a point P to move along the line AB from A towards B.

Because PA is zero when P is at A, therefore the ratio $\dfrac{PA}{AB}$ is also zero in that case.

Because PA increases continuously as P moves from A to B, therefore also the ratio $\dfrac{PA}{AB}$ increases continuously as P moves from A to B.

Because PA is equal to one half of AB when P is at O, therefore the ratio $\dfrac{PA}{AB}$ is equal to $\frac{1}{2}$ in that case,

therefore the ratio $\frac{PA}{AB}$ passes through all values between zero and $\frac{1}{2}$ as P moves from A to O.

But we have proved that the ratio $\frac{L}{L+M}$ has some value between zero and $\frac{1}{2}$,

therefore the ratio $\frac{PA}{AB}$ is equal to the ratio $\frac{L}{L+M}$ for one position of P between A and O.

Let this position be the point C.

Because $\frac{CA}{AB}$ is equal to $\frac{L}{L+M}$, therefore $\frac{CA}{AB-CA}$ is equal to $\frac{L}{M}$ (Bk. V. Prop. 9),

that is $\frac{CA}{CB}$ is equal to $\frac{L}{M}$,

therefore there is one point between A and O, such that the ratio of its distances from A and B respectively is equal to the given ratio $\frac{L}{M}$.

Next suppose the point P to move along AB produced towards A, then it may be proved by reasoning similar to the preceding that there is one position of P on this produced part of AB, such that the ratio $\frac{PA}{AB}$ is equal to the ratio $\frac{L}{M-L}$.

Let this position be the point D.

Because $\frac{DA}{AB}$ is equal to $\frac{L}{M-L}$,

therefore $\frac{DA}{AB+DA}$ is equal to $\frac{L}{M}$ (Bk. V. Prop. 9),

that is $\frac{DA}{DB}$ is equal to $\frac{L}{M}$,

therefore there is one point on the line AB produced towards

A, such that the ratio of its distances from A and B respectively is equal to the given ratio $\frac{L}{M}$.

If L be greater than M, then it may be proved by reasoning in all respects similar to that employed above that there are two points, one upon OB, and the other on AB produced towards B, such that the ratio of the distances of each of these points from A and B respectively is equal to the given ratio $\frac{L}{M}$.

DEFINITIONS.

64.—When a point is taken upon a given finite straight line, such that the ratio of its distances from the two extremities of the given line to one another is equal to a given ratio, the given straight line is said to be divided *internally* at that point in the given ratio.

65.—When a point is taken upon one of the produced portions of a given finite straight line, such that the ratio of its distances from the two extremities of the given line to one another is equal to a given ratio, the given straight line is said to be divided *externally* at that point in the given ratio.

Thus, in the last Proposition the straight line AB is divided *internally* at the point C, and *externally* at the point D, in the ratio $\frac{L}{M}$.

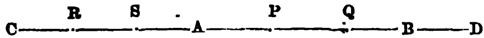
When a straight line is said to be divided in a given ratio, without further description, it is assumed that the division is internal.

Note.—It results from Proposition 6 that a straight line may always be divided in a given ratio in two points, in one point internally, and the other point externally. We now go on to prove that a straight line cannot be divided in a given ratio in more than two points.

PROPOSITION 7.

A given finite straight line cannot be divided in a given ratio in more than two points, viz., in one point internally, and in the other point externally.

Fig. 9.



Let AB be the given finite straight line, then

1st. It is not possible to divide AB *internally*, in any given ratio, in more than one point.

If possible, let P and Q be two points, each dividing AB internally in the same given ratio,

$$\text{therefore } \frac{AP}{PB} \text{ is equal to } \frac{AQ}{QB}.$$

$$\text{therefore } \frac{AP}{AP+PB} \text{ is equal to } \frac{AQ}{AQ+QB} \text{ (Bk. V. Prop. 9),}$$

$$\text{i.e. } \frac{AP}{AB} \text{ is equal to } \frac{AQ}{AB},$$

therefore AP and AQ have, each of them, the same ratio to AB.

Therefore AP is equal to AQ,

therefore the points P and Q coincide.

2nd. It is not possible to divide AB *externally* in more than one point in any given ratio.

If possible, let R and S be two points, each dividing AB externally in the same given ratio, and let them be situated on the same produced part of AB, as for instance on AC,

$$\text{therefore } \frac{AR}{BR} \text{ is equal to } \frac{AS}{BS},$$

$$\text{therefore } \frac{AR}{BR-AR} \text{ is equal to } \frac{AS}{BS-AS} \text{ (Bk. V. Prop. 9),}$$

$$\text{i.e. } \frac{AR}{AB} \text{ is equal to } \frac{AS}{AB},$$

therefore the points R and S coincide.

It is self-evident that the points R and S cannot divide AB externally in the same given ratio, when R and S are situated the one on the produced part AC, and the other on the produced part BD.

DEFINITIONS.

66.—When any four points are situated in the same straight line they are said to constitute a *range.*

67.—When the straight line bounded by two of these points is divided by the two remaining points, *externally* and *internally* in the same ratio, the range is called an *harmonic* range, and the straight line is said to be divided *harmonically.*

68.—The ratio which is equal to the quotient of the ratio of external division divided by the ratio of internal division is called the *anharmonic ratio* of the range.

Fig. 10.

Thus in Fig. 10, the points A, B, C, D, constitute a range.

If the ratio $\frac{AB}{AD}$ be equal to the ratio $\frac{CB}{CD}$ the range is called an harmonic range.

In all cases the ratio $\frac{AB}{AD}$ divided by the ratio $\frac{CB}{CD}$ is called the anharmonic ratio of the range.

Hence when the anharmonic ratio is unity, the range is an harmonic range.

Note.—Since $\frac{AB}{AD}$ divided by $\frac{CB}{CD}$ is equal to $\frac{AB}{AD}$ multiplied by $\frac{CD}{CB}$.

And since $\frac{AB}{AD}$ multiplied by $\frac{CD}{BC}$ is equal to $\frac{DC}{DA}$ multiplied by $\frac{BA}{BC}$, i.e. to $\frac{DC}{DA}$ divided by $\frac{BC}{BA}$, it follows that the anharmonic ratio of the range A, B, C, D, is the same whether we regard BD as divided externally and internally by A and C, or AC as divided externally and internally by B and D.

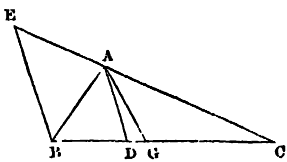
If the points B and D be fixed, and the points A and C move in such a way that A, B, C, D is always an harmonic range the points A and C are called *harmonic conjugates* with respect to the line BD, and it follows that the points B and D are also harmonic conjugates with respect to the line AC.

PROPOSITION 8.

If the vertical angle of a triangle be bisected by a straight line which also cuts the base, the segments of the base shall have to one another the same ratio that the adjacent sides of the triangle have, and if a straight line drawn through the vertical angle of a triangle divide the base into segments which have to each other the same ratio that the adjacent sides of the triangle have, then that line shall bisect the vertical angle of the triangle.

1st. Let the vertical angle BAC of the triangle ABC be bisected by the straight line AD which meets the base in D, then shall the ratio $\frac{BD}{DC}$ be equal to the ratio $\frac{BA}{AC}$.

Fig. 11.



Through B draw BE parallel to AD, then if CA and BE be produced they must meet in some point, let them meet in E.

Because EB is parallel to AD, therefore the angles AEB and ABE are equal to the angles CAD and BAD respectively (Bk. I. Prop. 19).

But by hypothesis BAD is equal to CAD,

therefore ABE is equal to AEB,

therefore EA is equal to BA.

Again, because AD is parallel to EB a side of the triangle CEB,

$$\text{therefore } \frac{EA}{AC} \text{ is equal to } \frac{BD}{DC}.$$

$$\text{But EA is equal to BA,}$$

$$\text{therefore } \frac{BA}{AC} \text{ is equal to } \frac{BD}{DC}.$$

2nd. Let $\frac{BD}{DC}$ be equal to $\frac{BA}{AC}$ then shall the angle BAC be bisected by AD.

For if not let, if possible, AG bisect the angle BAC.

Then it follows from what has been already proved that

$$\frac{BG}{GC} \text{ is equal to } \frac{BA}{AC}$$

$$\text{therefore } \frac{BG}{GC} \text{ is equal to } \frac{BD}{DC}$$

and this is impossible by Prop. 7.

Therefore no straight line other than AD can bisect the angle BAC.

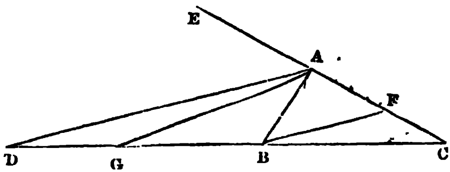
Therefore AD bisects the angle BAC.

PROPOSITION 9.

If the exterior angle of a triangle, made by producing one of its sides, be divided into two equal angles by a straight line which also cuts the base produced, then the base shall be divided externally in the point of intersection into segments which have to one another the same ratio that the adjacent sides of the triangle have; and if the base of a triangle be divided externally into segments which have to one another the same ratio that the sides of the triangle have, the line joining the point of division with the vertex of the triangle shall bisect the exterior angle at that vertex.

1st. Let the exterior angle BAE of the triangle ABC, made by producing the side CA to E, be bisected by the straight line AD which meets the base BC produced in D, then shall the ratio $\frac{DB}{DC}$ be equal to the ratio $\frac{BA}{AC}$.

Fig. 12.



Through B draw BF parallel to AD, and let BF meet AC in F.

Because AD is parallel to FB, therefore the angles DAB and DAE are equal to the angles ABF and AFB respectively (Bk. I. Prop. 19).

But by hypothesis DAB is equal to DAE;

therefore ABF is equal to AFB,
therefore AF is equal to AB,

Again, because BF is parallel to DA, a side of the triangle CDA, therefore $\frac{DB}{DC}$ is equal to $\frac{AF}{AC}$.

But AF is equal to BA,

therefore $\frac{DB}{DC}$ is equal to $\frac{BA}{AC}$.

2nd. Let $\frac{DB}{DC}$ be equal to $\frac{BA}{AC}$, then shall DA bisect the exterior angle BAE.

For if not let, if possible, AG bisect the angle BAC.

Then it follows from what has been already proved that

$$\frac{GB}{GC} \text{ is equal to } \frac{BA}{AC},$$

$$\text{therefore } \frac{GB}{GC} \text{ is equal to } \frac{DB}{DC},$$

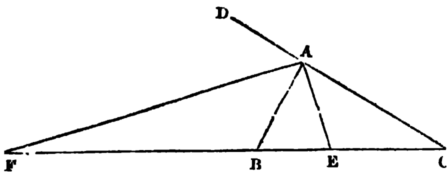
and this is impossible by Prop. 7.

Therefore no line other than AD can bisect the angle BAE.

Therefore AD bisects the angle BAE.

Corollary.—It follows from the two preceding Propositions, that if the interior and exterior angle between any two sides of a triangle be respectively bisected by two straight lines, each of which meets the third side of the triangle, then the two points of intersection of these lines with this third side will form, with the two remaining angular points of the triangle, an harmonic range.

Fig. 13.



For if the angles BAC and BAD, Fig. 13, be respectively bisected by the straight lines AE and AF, each of the ratios $\frac{BE}{EC}$ and $\frac{FB}{FC}$ is equal to $\frac{BA}{AC}$, and therefore $\frac{BE}{EC}$ is equal to $\frac{FB}{FC}$.

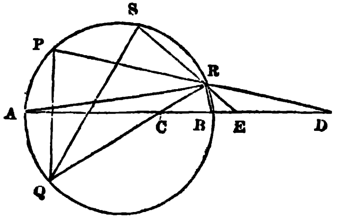
PROPOSITION 10.

If a triangle be inscribed in a circle and a diameter be drawn perpendicular to one of the sides, then the remaining two sides produced when necessary, shall divide this diameter harmonically; and, conversely, if two of the sides of an inscribed triangle divide any diameter of the circle harmonically, the third side shall be perpendicular to this diameter.

1st. Let ABQ be a circle, PQR any triangle inscribed in it, and AB the diameter perpendicular to PQ.

Let AB be divided internally in C and externally in D by the side QR and the side PR produced, then shall AB be divided harmonically in the points C and D.

Fig. 14.



Join AR and BR.

Because PQ is perpendicular to AB, therefore the arc AP is equal to the arc AQ,
therefore the angle ARP is equal to the angle ARQ, that is, AR bisects the angle PRQ.

Because ARB is a semicircle therefore ARB is a right angle.

Because the exterior angle CRP of the triangle CRD is bisected by RA, and RB is perpendicular to RA, therefore the interior angle CRD is bisected by RB.

Therefore the points A, C, B, and D form an harmonic range, or AB is divided harmonically in C and D (Bk. VI. Prop. 9, *Cor.*).

2nd. Let PQR be a triangle inscribed in the circle ABQ, and let the diameter AB be divided harmonically by the sides PR and QR in the points D and C, then shall PQ be perpendicular to AB.

For if not, let, if possible, QS be perpendicular to AB.

Join SR and produce it to meet AB produced in E.

By the first case it follows that AB is divided harmonically in C and E, or that $\frac{EB}{EA}$ is equal to $\frac{CB}{CA}$.

But by hypothesis $\frac{CB}{CA}$ is equal to $\frac{DB}{DA}$,

therefore $\frac{DB}{DA}$ is equal to $\frac{EB}{EA}$;

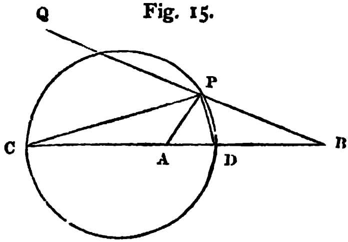
therefore the points D and E coincide (Bk. VI. Prop. 7),
therefore the points S and P coincide;
therefore PQ is perpendicular to AB.

PROPOSITION II.

The locus of a point whose distances from two fixed points are to each other in a given ratio is a certain circle.

Let A and B be two given points, and let D be a point on AB such that $\frac{AD}{DB}$ is equal to a certain given ratio, and let AD be less than DB.

Fig. 15.



Produce BA to C so that $\frac{CA}{CB}$ may be equal to $\frac{DA}{DB}$, which is always possible, as we know (Bk. VI. Prop. 6) then shall the circle described upon CD as diameter be the locus of a point whose distances from A and B are to each other in the given ratio $\frac{DA}{DB}$.

Let P be a point, not on BC, and such that $\frac{PA}{PB}$ is equal to the given ratio.

Join CP, AP, DP, and BP, and let BP be produced to Q.

Because $\frac{AP}{PB}$ is equal to $\frac{AD}{DB}$,

therefore PD bisects the angle APB.

And because $\frac{AP}{PB}$ is equal to $\frac{CA}{CB}$,

therefore PC bisects the exterior angle APQ ;

therefore CPD is a right angle ;

therefore P is a point on the circle whose diameter is CD.

Again, let P be a point on the circle whose diameter is CD.

Because the points C, A, D, and B, form an harmonic

range, therefore, by the last Proposition the angle QPA is bisected by CP;

$$\text{therefore } \frac{PA}{PB} \text{ is equal to } \frac{CA}{CB}.$$

But $\frac{CA}{CB}$ is equal to $\frac{DA}{DB}$,

$$\text{therefore } \frac{PA}{PB} \text{ is equal to } \frac{DA}{DB}.$$

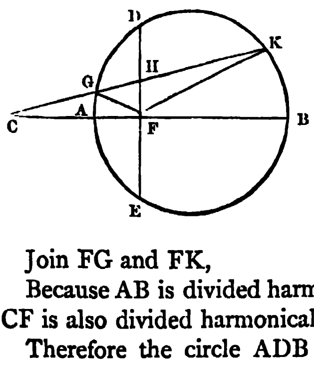
Therefore the locus of the required point is the circle upon CD as diameter.

PROPOSITION 12.

If a diameter of a circle be divided harmonically, and a line perpendicular to the diameter be drawn through the internal point of division, then any line drawn through the external point of division so as to cut the circle, shall meet the circle and the aforesaid perpendicular in three points, forming with the external point of division an harmonic range.

Let the diameter AB of the circle ADB be divided harmonically in the points C and F.

Fig. 16.



Through F the internal point of division, draw DFE perpendicular to AB, and through C draw CGHK, meeting the circle in G and K and the line DF in H.

Then shall C, G, H, and K form an harmonic range.

Join FG and FK,

Because AB is divided harmonically in C and F, therefore CF is also divided harmonically in A and B.

Therefore the circle ADB is the locus of points whose

distances from C and F are to each other in the ratio $\frac{AC}{AF}$ (last Prop.),

$$\text{therefore } \frac{GC}{GF} \text{ is equal to } \frac{KC}{KF};$$

$$\text{therefore } \frac{CG}{CK} \text{ is equal to } \frac{GF}{FK} \text{ (Bk. V. Prop. 9);}$$

therefore CF bisects the exterior angle between GF and KF (Bk. VI. Prop. 9).

But FD is at right angles to FC,

therefore FD bisects the interior angle between GF and KF;

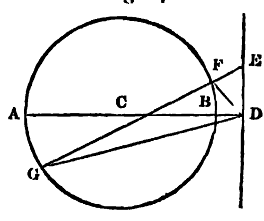
therefore C, G, H, and K form an harmonic range (Bk. VI. Prop. 9, *Cor.*).

PROPOSITION 13.

If a diameter of a circle be divided harmonically, and through the external point of division a straight line be drawn perpendicular to that diameter, then every straight line drawn through the internal point of division shall meet the circle and the aforesaid perpendicular in three points, forming, with the internal point of division, an harmonic range.

Let the diameter AB of the circle ABF be divided harmonically in the points C and D, and through D, the external point of division, draw DE at right angles to AD, and through C draw any straight line GCFE cutting the circle and the line DE in the points G, F, and E, then shall the four points G, C, F, and E form an harmonic range.

Fig. 17.



As in the last Proposition, we may prove that the circle ABF is the locus of points whose distances from C and D are to each other in the given ratio $\frac{BC}{BD}$;

$$\text{therefore } \frac{GC}{GD} \text{ is equal to } \frac{FC}{FD};$$

$$\text{therefore } \frac{GC}{FC} \text{ is equal to } \frac{GD}{FD};$$

therefore DC bisects the angle GDF.

But DE is at right angles to CD,

therefore DE bisects the exterior angle between GD and FD,

therefore G, C, F and E form an harmonic range.

DEFINITION.

69.—If the diameter of a circle be divided harmonically, the straight line drawn at right angles to this diameter through either point of division is called the *polar* of the other point of division with respect to the given circle.

Thus DE (Figs. 16 and 17) is the polar of the point C with respect to the circle ABF.

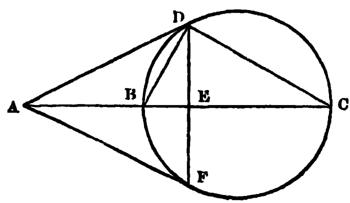
The point C is called the *pole* of the line DE.

PROPOSITION 14.

The polar of a point external to a given circle is the straight line which joins the points of contact of tangents to the circle from the given point.

Let A be the given point and BDC the given circle, draw AD and AF, touching the circle in D and F, and join DF, cutting AC, the line from A through the centre, at right angles in E.

Fig. 18.



Join DB and DC, then the angle DCB is equal to the angle ADB (Bk. II. Prop. 21).

But DCB is equal to BDE because each of these angles is the complement of the angle DBC,

therefore ADB is equal to BDE;

therefore BD bisects ADE and BDC is a right angle;
therefore DC bisects the angle between DE and AD produced;
therefore A, B, E, and C form an harmonic range, therefore DE is the polar of A.

EXAMPLES.

1. If the sides of a quadrilateral be bisected, prove that the lines joining the points of bisection taken in order form a parallelogram whose sides are parallel to the diagonals of the quadrilateral.

2. If AB be bisected in M and divided internally in C and externally in D into segments which have the same ratio, prove that AM or BM is a mean proportional to MC and MD.

3. If lines be drawn parallel to the base of a triangle, forming with the base and the segments of the sides a series of trapezoids, prove that the line joining the vertex with the middle point of the base will be the locus of the intersections of the diagonals of all these trapezoids.

4. ABC is an equilateral triangle and E a point on AC; on BC produced lengths CD and CF are taken respectively equal to CA and CE, and the lines AF and DE are drawn intersecting in H;

$$\text{prove that } \frac{HC}{EC} \text{ is equal to } \frac{AC}{AC+EC}.$$

5. AB and XY are any two straight lines; AB is divided at C in the ratio whose numerator and denominator are m and n respectively, and from the points A, B, and C, the lines AD, BE, and CF, are drawn perpendicular to XY;

prove that $(m+n)$CF is equal to nAD$+m$BE.

6. Prove that a straight line can only have one pole with respect to a given circle.

What is the pole of a tangent to the circle?

7. If any number of points be taken on a given straight line, prove that the polars of these points with respect to a given circle all pass through the pole of this line.

8. If any number of lines all pass through the same point, prove that the poles of these lines all lie on the polar of this point.

9.* If there be two luminous points whose powers of illuminating a point equally distant from each are in the ratio $\frac{m}{n}$, find the locus of a point in a plane containing them which is equally illuminated by them.

SECTION II.—ON SIMILAR RECTILINEAR FIGURES.

DEFINITIONS.

70.—Two polygons are said to be *similar* when—

1st. To every angle of the one there exists a corresponding equal angle of the other, and

2nd. The sides adjacent to any pair of corresponding angles of the one are proportional to the sides adjacent to the two corresponding angles of the other.

71.—A side of one of two similar polygons is said to be homologous to a side of the other, when the angles adjacent to the side of the one polygon correspond to the angles adjacent to the side of the other.

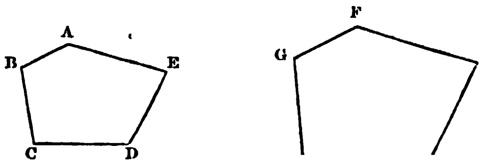
Thus if the two pentagons† ABCDE and FGHKL have the angles at A, B, C, D, and E respectively equal to the

* The illuminating power of a luminous point diminishes in the inverse square of the distance, i.e. if A be the luminous point, then the illumination produced by it at P is to that at Q as sq. of AQ is to sq. of AP.

† Polygons with five sides are called *pentagons*, with six sides *hexagons*, with seven sides *heptagons*, and so on, from the Greek words *pente*, *hex*, *hepta*, &c., signifying five, six, seven, &c.

angles at F, G, H, K, and L, and have also the ratios $\frac{AB}{BC}$, $\frac{BC}{CD}$, $\frac{CD}{DE}$, and $\frac{DE}{EA}$ respectively equal to the ratios $\frac{FG}{GH}$, $\frac{GH}{HK}$, $\frac{HK}{KL}$, and $\frac{KL}{LF}$, these pentagons will be similar.

Fig. 19.



The angles at A, B, C, D, and E are corresponding angles to the angles at F, G, H, K, and L respectively, and the sides AB, BC, CD, DE, and EA are homologous to the sides FG, GH, HK, KL, and LF respectively.

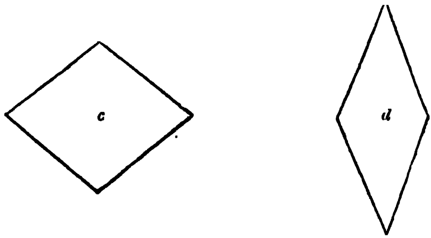
Note.—It is particularly important to remember that *two* conditions are required in order to satisfy the definition of similarity given above, *one* is the existence of corresponding equal angles in the two polygons, and *the other* is the proportionality of the homologous sides.

In every rectangular parallelogram all the angles are right angles, but all rectangular parallelograms are not on this account similar figures. For example, the ratio of the sides containing each of the angles of a square is a ratio of

equality, in all other rectangular parallelograms besides squares the adjacent sides are unequal.

Thus, the rectangular parallelograms *a* and *b* are *not* similar figures.

Again, the ratio of the sides containing every angle in any rhombus is a ratio of equality, but every rhombus is not on that account necessarily similar to every other rhombus, because the angles in the one are not necessarily equal to the angles in the other, each to each.



Thus each of the figures *c* and *d* is a rhombus, but they are *not* similar figures.

It will be proved hereafter that in the case of *triangles* the two conditions merge into one; or that one condition implies the other.

PROPOSITION 15.

If there be two similar polygons, the ratio of the perimeter of one polygon to the perimeter of the other shall be equal to the ratio of any side of one polygon to the side homologous to it of the other.

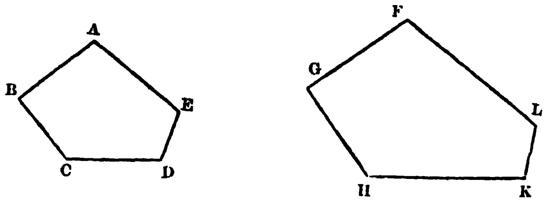
Let ABCDE and FGHKL be two similar pentagons, such that the angles at A, B, C, D and E correspond to the angles at F, G, H, K and L respectively, then shall the ratio $\frac{\text{perimeter of ABCDE}}{\text{perimeter of FGHKL}}$ be equal to the ratio $\frac{AB}{FG}$.

Because $\frac{AB}{BC}$ is equal to $\frac{FG}{GH}$,

and $\frac{BC}{CD}$ is equal to $\frac{GH}{HK}$,

therefore $\frac{AB}{FG}$ is equal to $\frac{BC}{GH}$ and $\frac{BC}{GH}$ is equal to $\frac{CD}{HK}$.

Fig. 20.



And similarly, each of the ratios $\frac{CD}{HK}$, $\frac{DE}{KL}$, and $\frac{EA}{LF}$ may be proved to be equal to the ratio $\frac{AB}{FG}$.

therefore $\frac{AB+BC+CD+DE+EA}{FG+GH+HK+KL+LF}$ is equal to $\frac{AB}{FG}$.

Therefore the perimeter of one pentagon is to the perimeter of the other, as a side of the one is to the side homologous to it of the other.

And similar reasoning may be applied to two similar polygons of any number of sides.

Corollary.—In two similar polygons the ratio of any one side of the one to the homologous side of the other, is equal to the ratio of any second side of the one to the side homologous to this second side of the other.

DEFINITION.

72.—In similar polygons the ratio of a side of the one to the side homologous to it of the other is called the *ratio of similitude* of the two polygons.

PROPOSITION 16.

If a line be arawn parallel to one side of a triangle, it will form, with the remaining sides or those sides produced, a second triangle similar to the first.

Fig. 21.

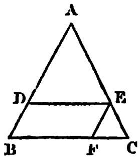


Fig. 22.

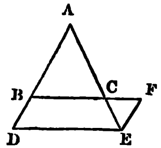
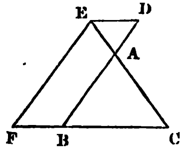


Fig. 23.



Let DE be parallel to BC, a side of the triangle ABC, then shall the triangle ADE be similar to the triangle ABC.

Through E draw EF parallel to AB, and let EF meet BC or BC produced in F, then BDEF is a parallelogram, and therefore DE is equal to BF.

Because DE is parallel to BC,

therefore the angles ADE and AED are equal to the angles ABC and ACB respectively (Figs. 21, 22, and 23),

therefore the triangles ABC and ADE are equiangular (Bk. I. Prop. 23, *Cor.*).

Again, because the lines AB and AC are cut by the two parallels DE and BC,

therefore $\frac{AD}{AB}$ is equal to $\frac{AE}{AC}$ (Bk. VI. Prop. 5, *Cor.*),

therefore $\frac{AD}{AE}$ is equal to $\frac{AB}{AC}$.

For a similar reason $\frac{AC}{AE}$ is equal to $\frac{BC}{BF}$, i.e. to $\frac{BC}{DE}$, since BF is equal to DE,

therefore $\frac{AC}{BC}$ is equal to $\frac{AE}{DE}$.

Because $\frac{AD}{AE}$ is equal to $\frac{AB}{AC}$, and $\frac{AE}{DE}$ is equal to $\frac{AC}{BC}$,

therefore $\frac{AD}{DE}$ is equal to $\frac{AB}{BC}$;

therefore the triangles ABC and ADE have corresponding equal angles, and the sides adjacent to these equal angles proportionals,

therefore ABC and ADE are similar triangles.

PROPOSITION 17.

If two triangles have two angles of the one respectively equal to two angles of the other, they shall be similar.

Fig. 24.

Let ABC and DEF be two triangles, having the angles at A and B equal to the angles at D and E respectively, then shall the triangle ABC be similar to the triangle DEF.

If AB be not less than DE, cut off AG equal to DE, and draw GH parallel to BC.

Because GH is parallel to BC, therefore the corresponding angles AGH and ABC are equal.

But ABC is equal to DEF,

therefore AGH is equal to DEF.

Because the angles at A and G and the side AG are respectively equal to the angles at D and E and the side DE, therefore the triangles AGH and DEF are equal in all their parts.

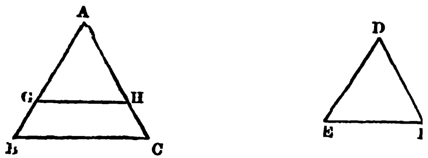
Because GH is parallel to BC, therefore the triangle ABC is similar to the triangle AGH,

therefore the triangle ABC is similar to the triangle DEF.

PROPOSITION 18.

If two triangles have an angle of the one equal to an angle of the other, and the sides containing those angles proportionals, the triangles shall be similar.

Fig. 25.



Let ABC and DEF be two triangles having the angles at A and D equal, and the ratio $\frac{AB}{AC}$ equal to the ratio $\frac{DE}{DF}$, then shall the triangle ABC be similar to the triangle DEF.

Make the same construction as in the last proposition.

Because GH is parallel to BC, therefore $\frac{AG}{AB}$ is equal to $\frac{AH}{AC}$.

But AG is equal to DE,

therefore $\frac{DE}{AB}$ is equal to $\frac{AH}{AC}$.

Because by hypothesis $\frac{DE}{DF}$ is equal to $\frac{AB}{AC}$, therefore $\frac{DE}{AB}$ is equal to $\frac{DF}{AC}$,

therefore $\frac{DF}{AC}$ is equal to $\frac{AH}{AC}$,

therefore DF is equal to AH;

therefore GA, AH and the angle GAH are respectively equal to ED, DF, and the angle EDF,

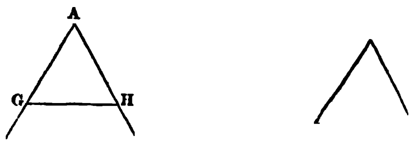
therefore the triangles AGH and DEF are equal in all their parts.

But the triangle AGH is similar to the triangle ABC, therefore the triangle ABC is similar to the triangle DEF.

PROPOSITION 19.

If two triangles have the sides about each of their angles proportionals they shall be similar.

Fig. 26.



Let ABC and DEF be two triangles having the ratios $\frac{AB}{BC}$, $\frac{BC}{CA}$ and $\frac{CA}{AB}$ equal to the ratios $\frac{DE}{EF}$, $\frac{EF}{FD}$ and $\frac{FD}{DE}$ respectively, then shall the triangle ABC be similar to the triangle DEF.

Make the same construction as in the two preceding propositions.

Because AGH and ABC are similar triangles,

therefore $\frac{AG}{AH}$ is equal to $\frac{AB}{AC}$.

But AG is equal to DE,

therefore $\frac{DE}{AH}$ is equal to $\frac{AB}{AC}$.

But by hypothesis $\frac{AB}{AC}$ is equal to $\frac{DE}{DF}$,

therefore also $\frac{DE}{AH}$ is equal to $\frac{DE}{DF}$,

therefore AH is equal to DF.

Again, because $\frac{AH}{HG}$ is equal to $\frac{AC}{CB}$, and $\frac{AC}{CB}$ is equal to $\frac{DF}{FE}$;

therefore $\frac{AH}{HG}$ is equal to $\frac{DF}{FE}$.

But AH is equal to DF;

therefore HG is equal to FE;

therefore AG, GH and HA are respectively equal to DE, EF, and FD,

therefore the triangles AGH and DEF are equal in all their parts.

But the triangle AGH is similar to the triangle ABC,

therefore the triangle ABC is similar to the triangle DEF.

Corollary.—It follows from the three preceding propositions that

1st. Two triangles in which the sides of the one are respectively parallel or perpendicular to the sides of the other are similar triangles,

2nd. Two right-angled triangles which have an acute angle of the one equal to an acute angle of the other, are similar triangles.

PROPOSITION 20.

If two straight lines be drawn through the same point so as to cut a circle, the rectangle of the segments of one of them shall be equal to the rectangle of the segments of the other.

Let the two straight lines AB and CD be drawn through the point E (Figs. 27 and 28) so as to cut the circle ADBC in the points A and B, C and D respectively, then shall the rectangle of AE and EB be equal to the rectangle of CE and ED.

Join AD and CB.

Because the angles DAB and DCB are in the same segment DACB,

therefore they are equal to one another.

Because AED and BEC are vertically opposite angles in Fig. 27, and coincident angles in Fig. 28, therefore they are equal to each other.

Fig. 27.

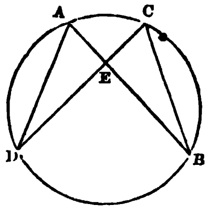
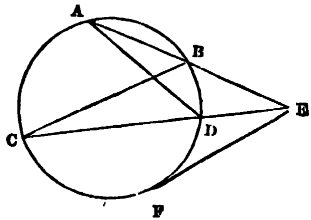


Fig. 28.



Because the triangles EAD and EBC have two angles of the one equal respectively to two angles of the other, therefore they are similar;

$$\text{therefore } \frac{AE}{ED} \text{ is equal to } \frac{CE}{EB};$$

therefore, the rectangle of AE and EB is equal to the rectangle of CE and ED (Bk. VI. Prop. 4).

Corollary.—If, in the case of Fig. 28, the straight line EDC be supposed to revolve round E in such a way as to increase the angle CEA, the points C and D will at length approach to coincidence in one point, F and EF will be a tangent to the circle; also, in this case, the rectangle of EC and ED becomes the square of EF, and therefore we have this proposition, viz.

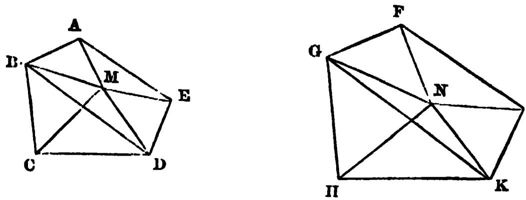
If from a point without a circle two straight lines be drawn, one of which cuts and the other touches the circle, then the rectangle of the segments of the cutting line shall be equal to the square of the touching line.

PROPOSITION 21.

If there be two similar polygons, and two points so situated that when the first point is joined to the extremities of a side of the one polygon, and the second to the extremities of the side

homologous to it of the other, the triangles thus formed are similar, and similarly situated with respect to each polygon, then all the triangles formed by joining each of these points with the extremities of any pair of homologous sides, one from each polygon respectively, shall be also similar, and similarly situated.

Fig. 29.



Let ABCDE and FGHKL be two similar pentagons, such that the sides AB, BC, &c., of the one are respectively homologous to the sides FG, GH, &c., of the other.

Also, let M and N be two points, such that the triangles MBC and NGH are similar and similarly situated with respect to each pentagon, i.e. either both of them on the same sides of the lines BC and GH as the pentagons are, or both on opposite sides of these lines, then the triangles formed by joining M and N with the extremities of any pair of homologous sides one from each pentagon shall be also similar, and similarly situated with respect to each pentagon.

Join M with D and N with K.

Because the angles BCD and BCM are by hypothesis equal to the angles GHK and GHN respectively,

therefore the angle MCD is equal to the angle NHK.

$$\left.\begin{array}{l}\text{Because } \dfrac{MC}{BC} \text{ is equal to } \dfrac{NH}{GH} \\ \text{and } \dfrac{BC}{CD} \text{ is equal to } \dfrac{GH}{HK}\end{array}\right\} \text{by hypothesis,}$$

$$\text{therefore } \frac{MC}{CD} \text{ is equal to } \frac{NH}{HK},$$

therefore the angles MCD and NHK of the triangles MCD and NHK are equal to each other, and the sides about these equal angles are proportionals;

therefore the triangles MCD and NHK are similar.

It is clear that they are also similarly situated with respect to the two pentagons.

In the same way it may be proved that the pairs of triangles formed by joining M and N with the extremities of successive pairs of homologous sides in each pentagon respectively, are pairs of similar and similarly situated triangles; and similar reasoning may be applied in the case of two similar polygons with any number of sides.

Such points as M and N are called *homologous points.*

Corollary 1.—Corresponding angular points, in the two polygons are also homologous points.

Corollary 2.—Two similar polygons may be divided into the same number of similar triangles, the vertices of the two sets of triangles being homologous points.

DEFINITION.

73.—When two straight lines are so drawn that the extremities of one of them are homologous to the extremities of the other, they are called *homologous lines.*

PROPOSITION 22.

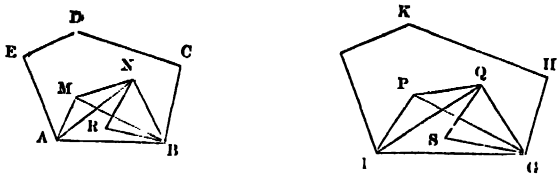
In two similar polygons the ratio of any two homologous lines is equal to the ratio of similitude of the two polygons, and if one pair of lines be homologous with another pair, the angle between the first pair shall be equal to that between the second pair.

Let ABCDE and FGHKL be a pair of similar pentagons in which AB and FG are a pair of homologous sides, and let M and N, P and Q be two pairs of homologous points,

so that the lines MN and PQ when joined are a pair of homologous lines, then shall

1st, the ratio $\frac{MN}{PQ}$ be equal to the ratio $\frac{AB}{FG}$.

Fig. 30.



Complete the construction as in the figure.

Because AMB and FPG are similar triangles, therefore the angles ABM and FGP are equal, and the ratio $\frac{MB}{AB}$ is equal to the ratio $\frac{PG}{FG}$.

Similarly, the angles ABN and FGQ are equal, and the ratio $\frac{AB}{NB}$ is equal to the ratio $\frac{FG}{QG}$.

Because the angles ABM and ABN are respectively equal to the angles FGP and FGQ,

therefore the angle MBN is equal to the angle PGQ.

Because $\frac{MB}{AB}$ is equal to $\frac{PG}{FG}$ and $\frac{AB}{NB}$ is equal to $\frac{FG}{QG}$,

therefore $\frac{MB}{NB}$ is equal to $\frac{PG}{QG}$.

Because MBN is equal to PGQ and $\frac{MB}{NB}$ is equal to $\frac{PG}{QG}$,

therefore the triangles MBN and PGQ are similar,

therefore $\frac{MN}{PQ}$ is equal to $\frac{MB}{PG}$.

Because the triangles AMB and FPG are similar,

$$\text{therefore } \frac{MB}{PG} \text{ is equal to } \frac{AB}{FG},$$

$$\text{therefore } \frac{MN}{PQ} \text{ is equal to } \frac{AB}{FG}.$$

2nd. Let R and S be another pair of homologous points, then shall the angle MNR be equal to the angle PQS.

It may be proved as before, that the triangles RNB and SQG are similar, and therefore that the angles RNB and SQG are equal.

But the angle MNB is equal to the angle PQG,

therefore the angle MNR is equal to the angle PQS.

A similar proof holds whatever be the number of sides.

Corollary.—If two similar polygons be so placed that any two sides of the one are parallel or perpendicular to the sides respectively homologous to them of the other, then every side of the one will be parallel or perpendicular to the side homologous to it of the other.

DEFINITION.

74.—When two similar polygons have their homologous lines all parallel, they are said to be *similarly situated.*

When the homologous lines are parallel, and the polygons on the same side of them, the polygons are said to be *similarly situated in the same direction*, and when the homologous lines are parallel, and the polygons upon opposite sides of them, they are said to be *similarly situated in opposite directions.*

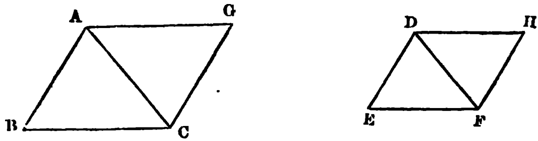
PROPOSITION 23.

The areas of two similar triangles are to one another in the square of the ratio of their homologous sides.

Let ABC and DEF be two similar triangles, having the angles at A, B, and C respectively equal to the angles at

D, E, and F, then shall the ratio $\frac{\text{area of ABC}}{\text{area of DEF}}$ be equal to the square of the ratio $\frac{BC}{EF}$.

Fig. 31.



Complete the parallelograms BG and EH, then the areas of BG and EH will be respectively double of the areas of ABC and DEF respectively.

Because BG and EH are equiangular parallelograms, therefore $\frac{\text{area of BG}}{\text{area of EH}}$ is equal to $\frac{AB}{DE} \times \frac{BC}{EF}$ (Bk. VI. Prop. 3).

Because ABC and DEF are similar triangles, therefore $\frac{AB}{DE}$ is equal to $\frac{BC}{EF}$,

therefore $\frac{\text{area of BG}}{\text{area of EH}}$ is equal to $\frac{BC}{EF} \times \frac{BC}{EF}$, i.e. to the square of $\frac{BC}{EF}$.

PROPOSITION 24.

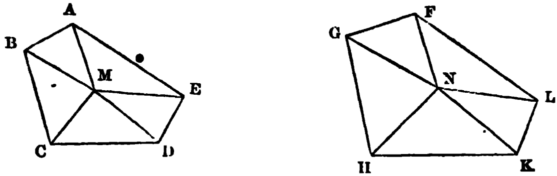
The areas of two similar polygons are to one another in the square of the ratio of their homologous sides.

Let ABCDE and FGHKL be two similar pentagons, of which AB and FG are homologous sides, then shall $\frac{\text{area of ABCDE}}{\text{area of FGHKL}}$ be equal to the square of the ratio $\frac{AB}{FG}$.

Let M and N be a pair of homologous points, and let them be joined with the angles in the respective pentagons as in the Figure, then the triangles MAB, MBC, &c., are

similar and equal in number to the triangles NFG, NGH, &c. (Bk. VI. Prop. 21, *Cor.* 2.).

Fig. 32.



Because MAB and NFG are similar triangles,

therefore $\frac{\text{area of MAB}}{\text{area of NFG}}$ is equal to the square of $\frac{\text{AM}}{\text{FN}}$ (Bk. VI. Prop. 23.)

Similarly, $\frac{\text{area of MAE}}{\text{area of NFL}}$ is equal to the square of $\frac{\text{AM}}{\text{FN}}$,

therefore $\frac{\text{area of MAB}}{\text{area of NFG}}$ is equal to $\frac{\text{area of MAE}}{\text{area of NFL}}$, and so on for each pair of corresponding triangles;

therefore $\frac{\text{sum of the areas of the triangles at M}}{\text{sum of the areas of the triangles at N}}$ is equal to

$\frac{\text{area of MAB}}{\text{area of NFG}}$, i.e. is equal to the square of $\frac{\text{AB}}{\text{FG}}$,

that is, $\frac{\text{area of ABCDE}}{\text{area of FGHKL}}$ is equal to the square of the ratio $\frac{\text{AB}}{\text{FG}}$.

A similar proof may be employed whatever be the number of sides of the two similar polygons.

Corollary.—The areas of two similar triangles or two similar polygons are to one another in the ratio of the squares of their homologous sides.

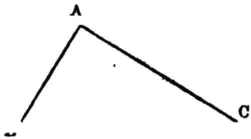
PROPOSITION 25.

If three similar polygons be described upon the three sides of a right-angled triangle, having the sides of the triangle for homologous sides of the polygons, the area of the polygon upon

the hypothenuse shall be equal to the sum of the areas of the similar polygons upon the other two sides.

Let ABC be a triangle right angled at A, and upon its three sides let three similar polygons be described, having the sides of the triangle for homologous sides of the polygon, then shall the area of the polygon upon BC be equal to the sum of the areas of the similar polygons upon AB and AC.

Fig. 33.



Because AB and BC are homologous sides of similar polygons,

therefore $\dfrac{\text{area of polygon on AB}}{\text{area of polygon on BC}}$ is equal to the ratio

$\dfrac{\text{square of AB}}{\text{square of BC}}$ (last Prop. *Cor.*).

Similarly, $\dfrac{\text{area of polygon on AC}}{\text{area of polygon on BC}}$ is equal to the ratio $\dfrac{\text{square of AC}}{\text{square of BC}}$,

therefore $\dfrac{\text{area of polygon on AB} + \text{area of polygon on AC}}{\text{area of polygon on BC}}$

is equal to $\dfrac{\text{square of AB} + \text{square of AC}}{\text{square of BC}}$.

But square of AB + square of AC is equal to square of BC, therefore the area of the polygon on AB + the area of the polygon on AC is equal to the area of the polygon on BC.

Examples.

1. If two triangles have one angle of the one equal to one angle of the other, and likewise one angle of the one supplementary to one angle of the other, the sides opposite to these angles respectively will be proportional.

2. If the sides of one triangle make respectively with the sides of another triangle, angles equal and in the same direction, the two triangles shall be similar.

3. Through A, one of the angles of the triangle ABC, a line AV is drawn parallel to the side BC, and another line ZXYV is drawn, cutting AB, BC, AC, and AV, or any of these lines produced in the points Z, X, Y and V respectively. Prove that VAZ and ZBX are similar triangles, and also VAY and YCX.

4. ABCD is a parallelogram, and the perpendiculars from its angular points upon the diagonals meet them in the points E, F, G, and H respectively. Prove that EFGH is a parallelogram similar to ABCD.

5. ABC is a right-angled triangle having the angle A the right angle. A line is drawn perpendicular to the hypothenuse cutting AB and AC produced if necessary in D and F. Prove that the locus of the intersection of CD and BF is the circle whose diameter is BC.

6. ABD is a circle of which AB is a diameter. From A the line AC is drawn perpendicular to AB, from any point C in AC the line CD is drawn touching the circle in D, DE is drawn perpendicular to AB and meeting AB in E, and BC is joined. Prove that DE is bisected by BC.

7. ABCD is a parallelogram, and P and Q are any two points in a line parallel to AB. If PA and QB meet in R, and PD and QC in S, prove that RS is parallel to AD.

8. Prove that the lines joining the angles of a triangle with the middle points of the opposite sides all pass through the same point, and also that this point divides each of the lines in the ratio of 2 to 1.

9. If perpendiculars be drawn upon any line whatever from the three angles of any triangle and the point found in the last example respectively, prove that sum of the three last perpendiculars will be one-third of the sum of the three first.

10. Prove that in every triangle the distance of the centre

of the circumscribing circle from any one of the sides is equal to half the straight line joining the angle opposite to that side with the point of intersection of the three perpendiculars drawn from the angles to the opposite sides.

11. If two similar figures be similar and similarly situated, prove that the lines joining any two homologous points in the two figures all pass through the same point.

12. If this point be called the centre of similitude, prove that the lines joining homologous points in similarly situated figures, are each divided by the centre of similitude into segments whose ratio is the ratio of similitude.

N.B.—The figures are said to be similarly situated in the same direction when the lines joining homologous points are divided externally in the centres of similitude, and to be similarly situated in opposite directions when these lines are divided internally in that point.

SECTION III.—ON REGULAR POLYGONS.

N.B.—The polygons treated of in this section are supposed to be concave polygons.

DEFINITION.

75.—A *regular polygon* is one which is both equilateral and equiangular.

PROPOSITION 26.

All regular polygons which have the same number of sides are similar.

Because the sum of the angles of any polygon, together with four right angles, is equal to twice as many right angles as the figure has sides, and because the two polygons have the same number of sides, therefore the sum of the angles

of the one polygon is equal to the sum of the angles of the other.

Because the polygons are regular, therefore every angle in each polygon is equal to the sum of the angles in each divided by the number of sides in each.

Therefore each angle of the one polygon is equal to each angle of the other.

And because the ratio of the sides about every angle of each polygon is a ratio of equality,

therefore the two polygons have corresponding equal angles, and the sides about these angles proportionals, therefore they are similar.

DEFINITIONS.

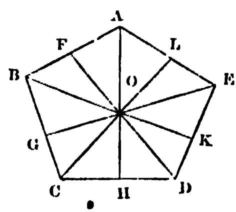
76.—A circle is said to be *described* or *circumscribed* about a polygon when every angular point of the polygon is upon the circle, and in this case the polygon is said to be inscribed in the circle.

77.—A circle is said to be *inscribed in* a polygon when each side of the polygon is a tangent to the circle, and in this case the polygon is said to be *described about* the circle.

PROPOSITION 27.

It is always possible to describe a circle about and to inscribe a circle in any regular polygon.

Fig. 34.



Let ABCDE be any regular pentagon, then

1st. It is always possible to describe a circle about ABCDE.

Bisect each of the angles at A and B by the straight lines AO, and BO, meeting at O, and join OE.

Because the two triangles OAB and OAE have the sides OA, AB, and the included angle

OAB of the one, respectively equal to the sides OA, AE, and the included angle OAE of the other, therefore they are equal in all their parts,

therefore the angle OBA is equal to the angle OEA.

But the angle CBA is double of the angle OBA;

therefore the angle CBA is also double of the angle OEA.

But the angle CBA is equal to the angle DEA;

therefore the angle DEA is double of the angle OEA.

And in the same way it may be shown that every angle of the pentagon is bisected by the line joining it with the point O.

Because the angles OAB and OBA are respectively halves of the equal angles EAB and ABC,

therefore the angle OAB is equal to the angle OBA,

therefore OA is equal to OB.

Similarly, all the lines from O to the angular points may be proved to be equal to one another.

Therefore the circle described with centre O and radius OA will pass through each of the angular points.

2nd. It shall be always possible to inscribe a circle in ABCDE.

Find the point O as in the last case, and from O draw OF and OG perpendicular to AB and BC respectively.

Because in the two triangles OBF and OBG the angles OBF and OFB are respectively equal to the angles OBG and OGB, and the side OB is common, therefore the triangles OBF and OBG are equal in all their parts.

Therefore OF is equal to OG.

Similarly, it may be proved that the perpendiculars drawn from O to each side of the pentagon in succession, are all equal to one another.

Therefore the circle described with O as centre, and radius OF, will pass through the feet of all these perpendiculars, and will touch each side in succession (Bk. II. Prop. 14).

A similar proof may be employed whatever be the number of sides in the two polygons.

Corollary 1.—The centres of the circles inscribed in and circumscribed about a regular polygon coincide.

Corollary 2.—If there be two regular polygons, the common centre of the inscribed and circumscribing circle of the one is homologous to the common centre of the inscribed and circumscribing circle of the other.

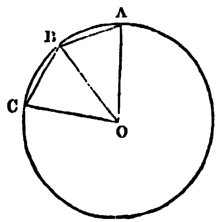
DEFINITION.

78.—The common centre of the inscribed and circumscribing circle is called *the centre* of the regular polygon.

PROPOSITION 28.

It is always possible to inscribe a regular polygon with any given number of sides in a given circle.

Fig. 35.



Let ABC be the given circle and O its centre.

Conceive the circumference to be divided into as many equal parts as there are sides in the required polygon in the points A, B, C, &c.

Join AB, BC, &c., thus forming an inscribed polygon with the required number of sides.

Because the arcs AB, BC, &c., are all equal,

therefore the chords AB, BC, &c., are also equal,

therefore the polygon formed by these chords is equilateral.

Again, because each of the angles of this polygon is subtended by an arc differing from the whole circumference by the same magnitude, viz. twice the arc AB,

therefore this polygon is equiangular (Bk. II. Prop. 11),

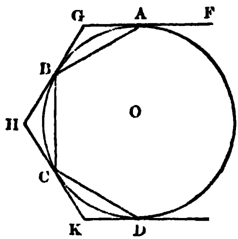
therefore it is a regular polygon of the required number of sides inscribed in the given circle.

PROPOSITION 29.

It is always possible to describe a regular polygon with any given number of sides about a given circle.

Let ABC be the given circle whose centre is O.

Fig. 36.



Conceive the circumference to be divided into as many equal parts as there are sides in the required polygon in the points A, B, C, &c.

Through A, B, C, &c., draw in succession tangents to the circle, viz. FG, GH, HK, &c., thus forming a polygon with the required number of sides circumscribing the given circle.

Join AB, BC, CD, &c.

Then each of the triangles GAB, HBC, KCD, &c., is an isosceles triangle (Bk. II. Prop. 20).

Because GBH touches, and BA and BC cut the circle, therefore the angles GBA and HBC are equal to the angles in the segments cut off by BA and BC respectively.

Because the arc BA is equal to the arc BC, therefore the angles in the segments cut off by BA and BC are equal,

therefore GBA is equal to HBC,

therefore the equal angles at B and A in the triangle GBA are equal to the equal angles at B and C in the triangle HBC, and the sides AB and BC are also equal,

therefore the triangles GBA and HBC are equal in all their parts,

therefore AGB is equal to BHC and HB is equal to BG.

Similarly, it may be proved that all the angles of the circumscribing polygon are equal to one another, and that the successive sides are bisected in the points of contact A, B, C, &c., respectively.

Because HB is equal to HC, and that HG and HK are double of HB and HC respectively,

therefore HG is equal to HK, and similarly for every other pair of sides,

therefore the polygon formed by the tangents is a regular polygon of the required number of sides.

Corollary 1.—It follows from the two preceding propositions, that if four right angles be divided into any number of equal parts and lines be drawn from the centre of a given circle inclined to each other successively at an angle equal to one of these equal parts, then the chords subtending these angles will form a regular polygon inscribed in the circle, and the tangents drawn through each of these points will form a regular polygon circumscribed about the circle, the number of sides of each polygon being equal to the aforesaid number of parts.

Corollary 2.—All regular polygons circumscribing a circle with the same number of sides are equal to each other in every respect, and similarly for regular inscribed polygons.

PROPOSITION 30.

If a regular polygon be inscribed in a given circle, and another regular polygon of the same number of sides as the former be described about the same circle, and if the number of sides of the polygons be increased indefinitely, remaining always equal in each, then the area of each of the polygons shall be ultimately equal to the area of the given circle.

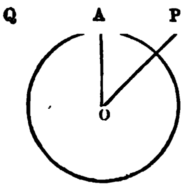
Let O be the centre of the given circle, and let A be one of the angular points of a regular polygon inscribed in the circle.

Let PAQ be one of the sides of a regular polygon of the same number of sides as the former polygon, but described about the given circle.

Join OA and OP.

Because OA and OP are homologous lines in the two polygons;

Fig. 37.



therefore $\frac{\text{area of described polygon}}{\text{area of inscribed polygon}}$ is equal to $\frac{\text{square of OP}}{\text{square of OA}}$.

Now let the number of sides be increased indefinitely, always remaining the same in each polygon, then the angle AOP will diminish indefinitely, the point P will approach to, and ultimately coincide with, the point A, and the square of OP will therefore become ultimately equal to the square of OA;

therefore ultimately the areas of the two polygons will become equal to each other;

therefore the area of the circle which is always intermediate between the areas of the polygons will become ultimately equal to the area of either of them.

PROPOSITION 31.

The circumferences of two circles are to each other as their radii, and the areas of two circles are to each other as the squares of their radii.

Let a regular polygon be inscribed in each circle, the number of sides in each polygon being the same.

Because the polygons are regular polygons, with the same number of sides in each, therefore they are similar;

therefore the ratio of the perimeters of the polygons to one another is equal to their ratio of similitude, and therefore equal to the ratio of the radii of the two circles.

Let the number of sides in each polygon be increased without limit, then the length of each side will be indefinitely diminished, and the perimeter of each polygon will ulti-

mately become equal to the circumference of its circumscribing circle (Note pp. 94 and 95),

therefore the circumferences of two circles are to one another as their radii.

Again, because the polygons are similar, therefore their areas are to one another in a ratio equal to the square of their ratio of similitude, and therefore equal to the ratio of the squares of the two radii.

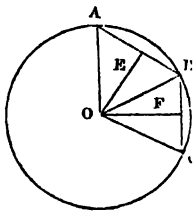
Let the number of sides be increased as before, and then the area of each polygon will ultimately become equal to the area of the circumscribing circle (last Prop.).

therefore the areas of two circles are to one another as the squares of their radii.

PROPOSITION 32.

The area of a circle is equal to half the rectangle of its radius and circumference.

Fig. 38.



Let ABC be a circle whose centre is O.

Let AB, BC, &c., be the sides of a regular polygon inscribed in the centre.

Let OE, OF, &c., be perpendiculars drawn from O to the sides AB, BC, &c., in succession, then OE, OF, &c., are all equal (Bk. II. Prop. 7).

Because the area of the polygon is equal to the sum of the areas of the triangles OAB, OBC, &c., therefore it is equal to one-half the sum of the rectangles of OE and AB, OF and BC, and so on.

But this latter sum is equal to the rectangle of OE and the sum AB+BC+&c. (Bk. IV. Prop. 11).

Now let the number of sides of the polygon be increased without limit, then the sum AB+BC+&c., becomes ultimately equal to the circumference of the circle, the line OE

becomes ultimately equal to the radius of the circle, and the area of the polygon becomes ultimately equal to the area of the circle;

therefore the area of the circle is equal to half the rectangle of its radius and circumference.

Examples.

1. If a quadrilateral be described about a circle, prove that the sums of the two pairs of opposite sides are equal to one another.

2. ABCDE is a regular pentagon, AC and BE are joined so as to intersect in F. Prove that CDEF is a parallelogram.

3. If every pair of alternate sides of a regular pentagon be successively produced to meet, prove that their points of intersection will be the angular points of another regular pentagon.

4. AB and CD are any two non-adjacent sides of a regular polygon, and E is its centre. Prove that if AB and CD can be produced to meet in F, then AFCE will be a quadrilateral inscribable in a circle.

5. Prove that a pavement may consist either of equilateral triangles, squares, or regular hexagons, and that it cannot consist of regular pentagons or of regular figures of more than six sides.

6. If an equilateral polygon be inscribed in a circle prove that it is a regular polygon; and so also if it be described about a circle, provided the number of sides be odd.

7. If a point be taken within a regular polygon, prove that the sum of its distances from the sides of the polygon is equal to the radius of the inscribed circle multiplied by the number of sides of the polygon.

8. Prove that the distance of the centre of any regular polygon from a straight line without it is the arithmetical

mean of the distances of the angular points of this polygon from this straight line.

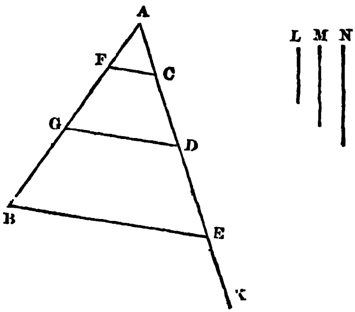
9. A semicircle is described upon a given straight line, and two semicircles are described upon the two halves of the line. Prove that if a circle be inscribed in the space between the three semicircles, its diameter will be two-thirds of that of each of the smaller circles.

Section IV.—Problems of Construction connected with Ratio and Proportion.

PROBLEM I.

To divide a given straight line into parts which shall be proportional to any given straight lines.

Fig 39.



Let AB be the given straight line to be divided, and L, M, N the given straight lines to which the parts of AB are to be proportional.

Through A draw an indefinite straight line AK, making any angle with AB, and cut off from AK the parts AC, CD, DE equal to L, M, and N respectively.

Join the point E with B, and draw CF and DG parallel to EB.

Because the straight lines AB and AE are cut by the series of parallels FC, GD, and BE,

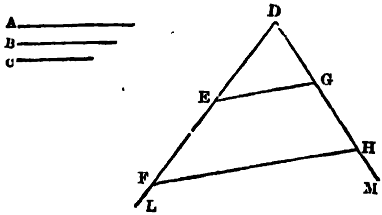
therefore the segments AF, FG, and GB are proportional to the segments AC, CD, and DE, i.e. to L, M, and N.

A similar construction will apply whatever be the number of lines L, M, N, &c.

PROBLEM 2.

To find the fourth proportional to three given straight lines.

Fig. 40.



Let A, B, and C be the three given straight lines, it is required to find the fourth proportional to A, B, and C.

Draw two indefinite straight lines DL and DM meeting at D, and from them cut off DE, EF, and DG equal to A, B, and C respectively.

Join EG and draw FH parallel to EG.

Because EG is parallel to FH,

$$\text{therefore } \frac{DE}{EF} \text{ is equal to } \frac{DG}{GH},$$

$$\text{therefore } \frac{A}{B} \text{ is equal to } \frac{C}{GH},$$

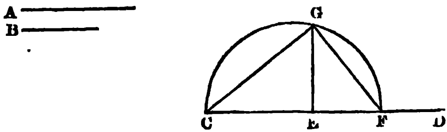
or GH is the fourth proportional required.

Corollary.—If C be equal to B the construction determines the third proportional to two given straight lines.

PROBLEM 3.

To find the mean proportional between two given straight lines.

Fig. 41.



Let A and B be the two given straight lines, it is required to find the mean proportional between A and B.

Take the indefinite straight line CD, and from it cut off CE and EF equal to A and B respectively.

Upon CF describe the semicircle CGF, and through E draw EG perpendicular to CD.

Join CG and FG.

Because CGF is a semicircle, therefore the angle CGF is a right angle.

Because CGF is a right angle, therefore the angle EGF is the complement of the angle CGE.

Because GEC is a right angle, therefore the angle GCE is the complement of the angle CGE;

therefore the angle GCE is equal to the angle EGF;
therefore the right-angled triangles CGE and FGE are similar (Bk. VI. Props. 17 to 19, *Cor.* 2),

$$\text{therefore } \frac{CE}{EG} \text{ is equal to } \frac{EG}{EF},$$

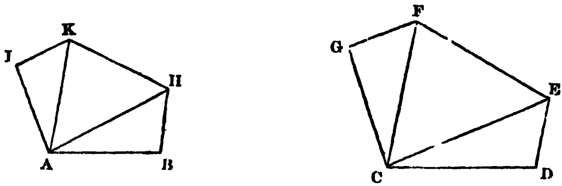
or EG is the mean proportional between CE and EF, i.e. between A and B.

Corollary.—If from the right angle of any right-angled triangle a perpendicular be drawn to the hypothenuse, this perpendicular will divide the triangle into two triangles similar to the whole and to one another.

PROBLEM 4.

To a given finite straight line to apply a polygon similar to a given polygon.

Fig. 42.



Let AB be a given finite straight line, and CDEFG a given pentagon; it is required to apply to AB a pentagon similar to CDEFG.

Join CE and CF.

At the points A and B make the angles HAB and HBA equal to the angles ECD and EDC respectively.

At the points A and H make the angles KAH and KHA equal to the angles FCE and FEC respectively.

At the points A and K make the angles LAK and LKA equal to the angles GCF and GFC respectively.

Because the triangles ABH and CDE have two angles of the one equal to two angles of the other, each to each, therefore they are similar,

therefore the angle AHB is equal to the angle CED, and the ratios $\frac{AB}{BH}$ and $\frac{BH}{AH}$ are equal to the ratios $\frac{CD}{DE}$ and $\frac{DE}{CE}$ respectively.

Similarly, the angle AKH is equal to the angle CFE, and the ratios $\frac{AH}{HK}$ and $\frac{HK}{AK}$ are equal to the ratios $\frac{CE}{EF}$ and $\frac{EF}{CF}$ respectively.

Because the angles AHB and AHK are equal to the angles CED and CEF respectively,

therefore the whole angles BHK and DEF are equal.

In like manner it may be shown that to every angle of the one pentagon ABHKL there is a corresponding equal angle of the other pentagon CDEFG.

Because $\frac{AH}{HK}$ is equal to $\frac{CE}{EF}$, and $\frac{BH}{AH}$ is equal to $\frac{DE}{CE}$;

therefore $\frac{BH}{HK}$ is equal to $\frac{DE}{EF}$.

And in like manner it may be shown that the sides about the angles of one pentagon are proportional to the sides about the corresponding angles of the other;

therefore the pentagon ABHKL is similar to the pentagon CDEFG, and it is applied to the straight line AB.

The same construction must be repeated for every additional side in the case of any polygon whatever.

PROBLEM 5.

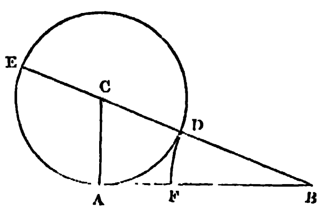
To divide a given straight line in extreme and mean ratio, i.e. so that one of the parts may be a mean proportional between the whole line and the remaining part.

Let AB be the given straight line.

Through A draw AC perpendicular to AB and equal to half of AB.

With centre C and radius CA describe the circle DAE.

Fig 43.



Join BC, meeting the circle in D, and produce it to meet the circle again in E.

With centre B and radius BD, describe the circular arc DF, cutting AB in F;

then F shall be the point of division required.

Because AB and DE are each double of CA, therefore they are equal to one another.

Because CAB is a right angle, therefore BA touches the circle DAE in A.

Because BA touches and BE cuts the circle,

therefore the square of BA is equal to the rectangle of BE and BD;

therefore $\frac{BE}{BA}$ is equal to $\frac{BA}{BD}$ (Bk. VI. Prop. 4);

therefore $\frac{BE-BA}{BA}$ is equal to $\frac{BA-BD}{BD}$;

But BA is equal to DE, and BD to BF;

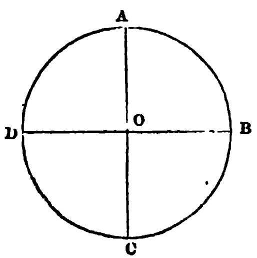
therefore $\frac{BD}{BA}$ is equal to $\frac{AF}{BD}$;

therefore $\frac{BF}{BA}$ is equal to $\frac{AF}{BF}$.

PROBLEM 6.

To inscribe a square in a given circle, and to describe a square about a given circle.

Fig. 44.



Let ABCD be the given circle whose centre is O; it is required to inscribe a square in, and to describe a square about ABCD.

Draw the diameters AOC and BOD at right angles to each other, and cutting the circle in the points A, B, C, and D.

If the chords ABCD be joined, the quadrilateral so formed will be an inscribed square.

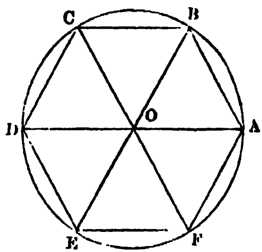
Also if tangents be drawn through the points A, B, C, D, the quadrilateral so formed will be a described square.

The proof is evident from Bk. VI. Prop. 29, *Cor.* 1.

PROBLEM 7.

To inscribe a regular hexagon in, and to circumscribe a regular hexagon about, a given circle.

Fig. 45.



Let ABC be the given circle whose centre is O; it is required to inscribe a regular hexagon in, and to describe a regular hexagon about, the circle ABC.

With centre A, any point on the circle, and radius AO, describe a circular arc cutting the circle in B; then if the straight lines AB, OB, and OA be joined they will be all equal.

Because OAB is an equilateral triangle, it is also equiangular;

therefore the angle AOB is the third part of two right angles;

therefore AOB is the sixth part of four right angles.

Draw through O the lines OC, OD, OE, and OF, making the angles BOC, COD, DOE, and EOF, each equal to BOA, or one-sixth part of four right angles, then the remaining angle FOA will be also equal to one-sixth part of four right angles.

If the chords BC, CD, DE, EF, and FA be drawn, the six-sided figure thus formed will be a regular hexagon inscribed in the circle.

And if tangents be drawn through the points A, B, C, D, E, and F, the six-sided figure thus formed will be a regular hexagon described about the circle.

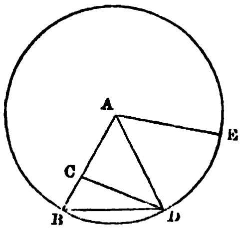
The proof is evident from Bk. VI. Prop. 29, *Cor.* 1.

PROBLEM 8.

To inscribe a regular decagon in, and to describe a regular decagon about, a given circle.

Let BDE be the given circle; it is required to inscribe a regular decagon in, and to describe a regular decagon about, BDE.

Fig. 46.



Take any radius, AB, and divide it in C in extreme and mean ratio, so that $\frac{BA}{AC}$ may be equal to $\frac{AC}{CB}$.

With centre B and radius equal to AC, describe a circular arc, cutting BDE in D.

Join BD, CD, and AD.

Because $\frac{BA}{AC}$ is equal to $\frac{AC}{CB}$, and BD is equal to AC;

therefore $\frac{AB}{BD}$ is equal to $\frac{DB}{BC}$;

therefore ABD and DBC are similar triangles (Bk. VI. Prop. 18);

therefore BDC is equal to BAD.

Again, because $\frac{BA}{AC}$ is equal to $\frac{AC}{CB}$, and AB and AC are equal to AD and BD respectively,

therefore $\frac{AD}{DB}$ is equal to $\frac{AC}{CB}$,

therefore BDC is equal to CDA (Bk. VI. Prop. 8).

Because BDC is equal to CDA, therefore BDA is double of BDC.

But BDC is equal to BAD,

therefore BDA is double of BAD;

therefore also ABD is double of BAD,

therefore ABD+ADB+BAD is equal to 5 times BAD;

therefore BAD is the fifth part of two right angles;

therefore BAD is the tenth part of four right angles.

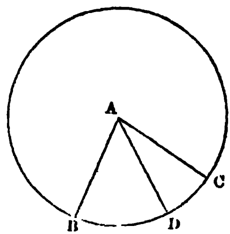
The remainder of the construction is obvious from (Bk. VI. Prop. 29, *Cor.* 1).

Corollary.—If DAE be taken equal to BAD the angle EAB will be the fifth part of two right angles, and hence we obtain the construction for inscribing a regular pentagon in, or describing such a pentagon about, a given circle.

PROBLEM 9.

To inscribe a regular quindecagon in, and to describe a regular quindecagon about, a given circle.

Fig. 47.



Let BDC be the given circle whose centre is A.

Draw any radius, AB, and draw the radii AC and AD, making with AB the angles BAC and BAD equal to the sixth and tenth part of four right angles respectively, by Problems 6 and 7.

Then, if four right angles were divided into thirty equal parts, BAC would contain five, and BAD three of these parts respectively,

therefore CAD contains two of these parts,

therefore CAD is the fifteenth part of four right angles.

The remainder of the construction is obvious from Prop. 29, *Cor.* 1.

Note.—The two following Problems are chosen for the purpose of illustrating a method which is frequently of great use in arriving at geometrical constructions.

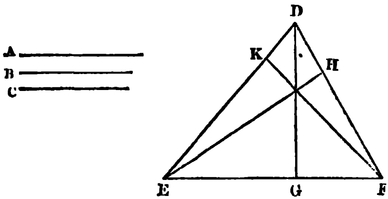
The principle of this method consists in dividing the problem into two parts, each of which taken separately is often easy of solution.

One of these parts is the construction of a figure *similar* to some required figure, and the other the determination of the ratio of similitude between this figure and the required figure.

PROBLEM 10.

To construct a triangle such that the three perpendiculars from the angles upon the opposite sides may be respectively equal to three given straight lines.

Fig. 48.



Let A, B, and C be the three given straight lines; it is required to construct a triangle such that the perpendiculars from the angles upon the opposite sides may be respectively equal to A, B, and C.

Let DEF be any triangle, and DG, EH, and FK the three perpendiculars from the angles upon the opposite sides.

Because the right-angled triangles DFG and EFH have a common acute angle at F, therefore they are similar;

$$\text{therefore } \frac{DF}{DG} \text{ is equal to } \frac{EF}{EH}.$$

Similarly, $\frac{ED}{EH}$ is equal to $\frac{FD}{FK}$,

$$\text{and } \frac{DE}{DG} \text{ is equal to } \frac{FE}{FK}.$$

Because $\frac{DF}{FK}$ is equal to $\frac{DE}{EH}$,

$$\text{therefore } \frac{DF}{DE} \text{ is equal to } \frac{FK}{EH}.$$

But $\frac{DF}{DG}$ is equal to $\frac{DF}{DE} \times \frac{DE}{DG}$,

$$\text{therefore } \frac{DF}{DG} \text{ is equal to } \frac{FK}{EH} \times \frac{DE}{DG}.$$

Let L be a fourth proportional to KF, EH, and DG,

so that $\frac{DG}{L}$ is equal to $\frac{FK}{EH}$,

therefore $\frac{DF}{DG}$ is equal to $\frac{DG}{L} \times \frac{DE}{DG}$;

therefore $\frac{DF}{DG}$ is equal to $\frac{DE}{L}$;

also $\frac{DF}{DG}$ is equal to $\frac{EF}{EH}$;

therefore the ratios $\frac{DE}{L}$, $\frac{DF}{DG}$, and $\frac{EF}{EH}$ are all equal;

therefore the triangle DEF is similar to the triangle whose sides are L, DG, and EH,

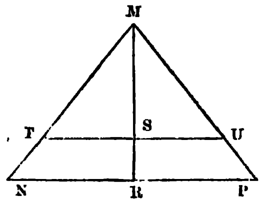
whence we have this construction.

Find the fourth proportional L to the three lines A, B, and C.

Construct the triangle whose sides are A, B, and L.

Then this triangle will be similar to the required triangle.

Fig. 49.



Again, to find the ratio of similitude.

Let MNP be the triangle thus constructed, i.e. such that PM is equal to A, MN to B, and NP to L.

From M draw MR perpendicular to NP.

If MR be equal to C, then MPN will be the required triangle.

If not, $\frac{C}{MR}$ will be the ratio of similitude in question.

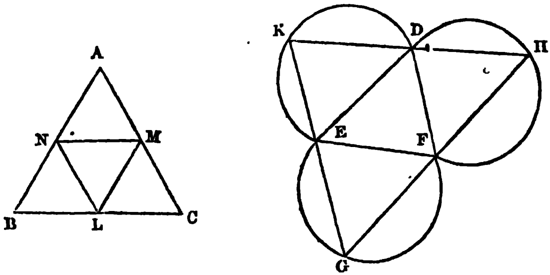
PROBLEM II.

In a given triangle to inscribe a triangle similar to a given triangle.

Let ABC and DEF be two given triangles; it is required to inscribe in ABC a triangle similar to DEF.

Upon EF, FD, and DE, describe segments of circles EGF, DHF, and DKE, containing angles equal to A, B, and C

Fig. 50.



respectively, through D draw any straight line, HDK, and join HF and KE respectively.

Because the angles at H and K are equal to ABC and ACB respectively, therefore the straight lines HF and KE must be inclined to each other at an angle equal to A;

therefore KE and AF, if produced, must intersect in some point G upon the segment EGF;

therefore the triangle GHK is similar to ABC, and circumscribes the triangle DEF.

Hence the first part of the construction is completed.

If the sides BC, AC, and AB of the triangle ABC be divided in L, M, and H in the same ratio as the sides of the triangle GHK are divided in D, E, and F, it may be easily proved that the triangle LMN is similar to the triangle DEF.

Hence the construction is completed.

It is clear that GHK is only one of an infinite number of triangles which might be drawn about DEF similar to ABC.

Therefore LMN is only one of an infinite number of triangles which might be inscribed in ABC similar to DEF.

EXAMPLES.

1. Describe a triangle similar to a given triangle and having a given perimeter.

2. Through a given point draw a straight line, so that the segment intersected thereupon by two given straight lines may be divided by that point in a given ratio.

3. In a given triangle inscribe a parallelogram similar to a given parallelogram.

4. In one of two given triangles inscribe a triangle having its sides parallel to the sides of the other triangle.

5. In a given triangle inscribe the least possible triangle similar to a given triangle.

6. Through a given point draw a straight line so that the part of it cut off by a given circle may be divided at that point in a given ratio.

7. Divide a given arc of a circle into two parts which shall have their chords in a given ratio.

8. Construct a triangle so that one side and the sum of the two other sides may be each equal to a given magnitude, and the angle opposite the given side may be on a given straight line.

9. Given three points A, B, and C; draw a straight line through A so that the distances from A to the feet of the perpendiculars upon it from B and C may be in a given ratio.

10. Given two finite straight lines; draw through a given point a straight line which would pass through their point of intersection, without producing the lines.

11. Through one of the points of intersection of two given circles draw a straight line such that the chords intercepted on it by the two circles may be in a given ratio.

12. Describe a circle passing through two given points and touching either a given straight line or a given circle.

13. Through a given point describe a circle touching two given straight lines.

14. Through a given point describe a circle so as to touch a given straight line and a given circle.

15. Describe a circle touching two given straight lines and a given circle.

16. Describe a circle touching a given straight line and two given circles.

17. Describe a circle touching three given circles.

Miscellaneous Examples.

1. ABCD is a quadrilateral figure inscribed in a circle, and AC and BD are its diagonals intersecting in E.

Prove that $\frac{AB}{AD} \times \frac{BC}{DC}$ is equal to $\frac{BE}{DE}$.

2. OABC is a rhombus whose sides OA and OC are produced indefinitely. In OA and OC, any two points, P and Q, are taken, and CP and AQ are joined, cutting AB and BC in the points D and E respectively. Prove that DE is parallel to PQ.

3. Apply this proposition to determine the distance between P and Q, supposing them to be two inaccessible objects situated on the ground, and to trace a line on the ground parallel to PQ.

4. ABC is a right-angled triangle, of which BAC is the right angle. A square DEGF is inscribed in the triangle in such a way that DE coincides with BC. Prove that DE is a mean proportional between BD and EC.

5. ABC is a triangle inscribed in a circle and through B, the line BD is drawn parallel to the tangent to the circle at A, meeting AC produced if necessary in D, prove that AB is a mean proportional to AC and AD.

6. On the side BC of the triangle ABC, and on the side opposite to the triangle ABC the square BCDE is described, and AE and AD are joined, cutting BC in the points P and Q. Prove that PQ is equal to the side of the square inscribable in the triangle ABC, and having one of its sides coincident with BC.

7. Any point is taken on an arc of a circle. Prove that the distance of this point from the chord of the arc is a mean proportional between the distances of the same point from the tangents at the extremities of the arc.

8. If a tangent be drawn to any circle at any point, prove that the radius of the circle is a mean proportional between the lengths of this tangent intercepted between the point of contact and the tangents at the extremities of any diameter.

9. ABC is a triangle, and M is the middle point of the side BC. I is the point in which the inscribed circle touches BC, and H and K are the points in which the perpendicular from A upon BC, and the line bisecting the angle at A meet BC. Prove that

$$\frac{MI}{MH} \text{ is equal to } \frac{IK}{IH}.$$

10. Given an angle of a triangle, the perpendicular from this angle upon the opposite side, and the sum or difference of the remaining sides. Construct the triangle.

11. Through two given points draw two straight lines intersecting on a given circle, and such that the chord joining the other two points of intersection with the circle may be parallel to the straight line joining the two given points.

12. Determine a point such that its distances from the three sides of a given triangle shall be proportional to three given straight lines.

13. Divide a given straight line into three parts, such that the first and second may be in a certain given ratio, and the second and third in another given ratio.

14. If from the angles of an equilateral triangle perpendiculars be drawn upon any diameter of the circumscribing circle, prove that the perpendicular which falls upon one side of that diameter will be equal to the sum of the two perpendiculars which fall upon the other side.

15. In a given square inscribe four equal circles touching one another.

BOOK VII.

ON PLANES, AND LINES IN SPACE.

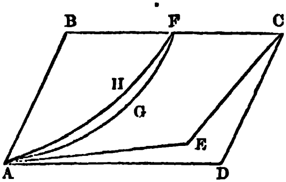
SECTION I.—MISCELLANEOUS PROPOSITIONS.

PROPOSITION I.

It is always possible to find one plane, and only one plane, containing a given straight line and a given point not in that straight line.

Let BC be the given straight line and A the given point not situated in the straight line BC, then it shall be always possible to find one plane, and only one plane, containing the straight line BC and the point A.

Fig. 1.



Suppose a plane of indefinite extent to revolve round the straight line BC, then since in the course of its revolution such a plane must pass through every point in space, therefore there must be some one position of this plane in which it passes through the given point A.

That is, one plane may always be found containing the given straight line BC and the given point A.

Also, no more than one such plane can be found.

If possible, let ABCD and ABCE be two non-coincident planes, each of which contains the straight line BC and the point A.

Take any point F in BC.

Because each of the points A and F is situated in the

plane ABCD, therefore a straight line AGF may be drawn from A to F, and situated in the plane ABCD (Def. 9).

Similarly, a straight line AHF may be drawn from A to F, and situated in the plane ABCE.

Therefore the two non-coincident straight lines AGF and AHF intersect in two points A and F,

which is impossible (Ax. 1);

therefore the planes ABCD and ABCE must coincide in every point.

Corollary 1.—One plane, and only one plane, may always be found containing three given points not situated in the same straight line.

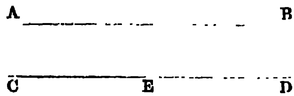
Corollary 2.—One plane, and only one plane, may always be found containing two given intersecting straight lines.

PROPOSITION 2.

It is always possible to find one plane, and only one plane, containing two given parallel straight lines.

Let AB and CD be the two given parallel straight lines, then it shall be always possible to find one plane, and only one plane, containing both of the straight lines AB and CD.

Fig. 2.



Because AB and CD are parallel straight lines, therefore they are situated in one plane (Def. 27);

that is, one plane may always be found containing both AB and CD.

Also no more than one such plane can be found.

For let E be any point in CD.

Because E is a point not situated in the straight line AB, therefore only one plane can be found containing the straight line AB and the point E (Bk. VII. Prop. 1);

therefore only one plane can be found containing the two given parallel straight lines AB and CD.

Note.—It follows from the two preceding propositions that a plane is completely determined in each of the three following cases:

1. When it contains a given straight line and a given point not in that line.

2. When it contains three given points not situated in the same straight line.

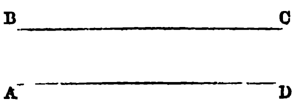
3. When it contains each of two given intersecting or parallel straight lines.

PROPOSITION 3.

Through any given point it is always possible to draw one straight line, and one straight line only, parallel to any given straight line not containing the given point.

Let A be the given point, and BC the given straight line not containing the point A, then it shall be always possible to draw one straight line, and one straight line only, through the point A parallel to the straight line BC.

Fig. 3.



Because the point A is not situated upon the straight line BC, therefore one plane, and one plane only, may always be found containing the straight line BC and the point A.

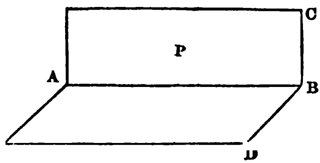
Because (*Ax.* 4 and Bk. III. Prob. 6) it is always possible to draw one straight line, and one straight line only, in the plane containing the straight line BC and the point A, and parallel to BC, let this straight line be AD; therefore AD will be drawn through the given point A, parallel to the given straight line BC, and will be the only straight line that can be so drawn.

PROPOSITION 4.

If two planes cut one another their line of intersection shall be a straight line.

Let ABC and ABD be two planes which cut one another, and let A and B be two points in their line of intersection, then if AB be joined, every point common to both of the planes shall be situated in the straight line AB or in AB produced.

Fig. 4.



If possible let P be a point in the line of intersection of these planes and not situated in the straight line AB or that line produced.

Because A, P, and B are three points not situated in the same straight line, therefore no more than one plane can be found which contains the points A, P, and B.

Because A, P, and B are three points situated in the line of intersection of the two planes ABC and ABD, therefore there are two planes ABC and ABD, each of which contains the three points A, P, and B.

But we have before proved that only one plane can be found containing these three points,

which is impossible;

therefore every point in the line of intersection of the planes ABC and ABD must be in the straight line AB;
that is, the line of intersection of these planes is a straight line.

DEFINITIONS.

79.—A straight line and a plane are said to be *parallel* to one another when they cannot meet, however far they may be produced in any direction whatever.

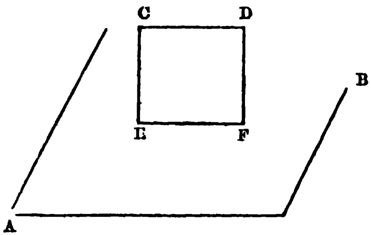
80.—Two planes are said to be *parallel* to one another when they cannot intersect, however far they may be produced in any direction whatever.

PROPOSITION 5.

If a given straight line be parallel to a given plane, every plane which contains the given straight line shall intersect the given plane in a straight line parallel to the given straight line.

Let AB be the given plane and CD the given straight line parallel to AB, and let CF, any plane containing the straight line CD, intersect AB in EF, then shall EF be parallel to CD.

Fig. 5.



Because CD is parallel to the plane AB therefore CD and the plane AB cannot meet, however far they may be produced in any direction whatever, therefore CD and EF cannot meet, however far they may be produced in either direction.

But CD and EF are by hypothesis in the same plane, therefore EF is parallel to CD (Def. 27).

PROPOSITION 6.

If a given straight line be situated in a given plane, every line parallel to the given straight line shall either be situated in the given plane or be parallel to that plane.

Take the same figure as in the last Proposition.

Let EF be the given straight line and AB the given plane in which EF is situated, and let CD be a straight line

parallel to EF, then shall CD either be situated in the plane AB or be parallel to that plane.

Because CD and EF are two parallel straight lines, therefore they are situated in the same plane.

If this plane be AB then the proposition is proved, but if not, let CF be the plane which contains both CD and EF, then EF is the line of intersection of the planes CF and AB.

Because the straight line CD is situated in the plane CF, and this plane intersects the plane AB in the line EF, therefore every point which is common to the line CD, and the plane AB must be also common to the line CD and the line EF.

Because CD and EF are parallel therefore they have no point in common;

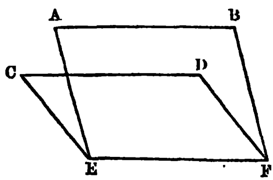
therefore the line CD and the plane AB cannot intersect, however far they may be produced,

therefore the line CD is parallel to the plane AB.

PROPOSITION 7.

If there be two planes of which one contains one and one the other of two given parallel straight lines, then the line of intersection of these planes shall be parallel to each of the two given straight lines.

Fig. 6.



Let AB and CD be two given parallel straight lines, and let AF and CF be any two planes containing AB and CD respectively, then EF the line of intersection of these planes shall be parallel to each of the lines AB and CD.

Because the straight line CD is situated in the plane CF, and that AB is parallel to CD, therefore AB is also parallel to CF (Bk. VII. Prop. 6).

Because the straight line AB is parallel to the plane CF, and that EF is the line of intersection of the plane AF containing AB, with the plane CF,

therefore AB is parallel to EF (Bk. VII. Prop. 5).

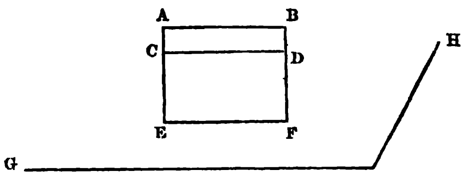
Similarly, CD is parallel to EF.

Corollary.—If two straight lines A and B be each of them parallel to a third straight line C they shall be parallel to one another. For the plane which contains A and some point P in B must intersect the plane containing B and C in the straight line B, otherwise through P two intersecting straight lines might be drawn each parallel to C, therefore B is parallel to A.

PROPOSITION 8.

If a straight line be parallel to a given plane, every straight line parallel to the given straight line shall be either situated in the given plane or be parallel to that plane.

Fig. 7.



Let the straight line AB be parallel to the plane GH, and let CD be another straight line parallel to AB, then CD shall be either situated in the plane GH or be parallel to that plane.

Because AB is parallel to CD, therefore a plane may be found which contains both AB and CD:

Let this plane be AF, and let AF intersect the plane GH in the straight line EF.

Because the straight line AB is parallel to the plane GH,

and that EF is the line of intersection of a plane which contains AB with the plane GH,

therefore EF is parallel to AB (Bk. VII. Prop. 5).

Because CD and EF are each of them parallel to AB, therefore CD is parallel to EF (Bk. VII. Prop. 7, *Cor.*).

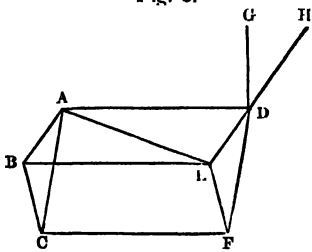
Because CD is parallel to EF, a straight line in the plane GH, therefore CD is either parallel to the plane GH or is coincident with it (Bk. VII. Prop. 6).

PROPOSITION 9.

If two intersecting straight lines be respectively parallel to and in the same direction with two other intersecting straight lines, the angle contained by the first pair of lines shall be equal to the angle contained by the second pair, and the planes in which these angles are situated shall be either coincident or parallel.

Let the two straight lines AB and AC which intersect in A be respectively parallel to and in the same direction with the two straight lines DE and DF which intersect in D, then shall the angle BAC be equal to the angle EDF, and the plane containing AB and AC shall be coincident with, or parallel to, the plane containing DE and DF.

Fig. 8.



If the straight lines AB and AC, DE and DF lie in the same plane, then the angles BAC and EDF are equal by Prop. 21, Bk. I., and the planes of these angles being coincident the proposition is proved.

But if the plane containing BAC be not coincident with that containing EDF, let AB, AC, DE, and DF be all equal to one another, and join AD, BE, CF, BC, EF, and AE.

Because AE meets the parallels AB and DE, therefore the angles BAE and AED are equal.

Because BA, AE, and the angle BAE are equal to DE, EA, and the angle DEA respectively,
therefore the triangles BAE and DEA are equal in all their parts; and therefore AD is equal to BE, and DAE is equal to BEA.

Because the alternate angles DAE and BEA are equal, therefore AD is parallel to BE.

Similarly, CF is equal and parallel to AD or BE (Prop. 7, *Cor.*).

Because CF is equal and parallel to BE, it may be proved, as above, that BC is equal and parallel to EF.

Because in the two triangles ABC and DEF the sides AB, AC, and BC of the one are respectively equal to the sides DE, DF, and EF of the other, therefore the angle BAC is equal to the angle EDF.

Again, because AB is parallel to DE, therefore AB is parallel to the plane DEF,
therefore if the plane BAC meet the plane DEF the line of intersection must be parallel to AB.

Similarly, it must be parallel to AC,
therefore from the point A two straight lines can be drawn, each parallel to the same straight line,
which is impossible;
therefore the plane ABC cannot intersect the plane DEF;
that is, the plane ABC is parallel to the plane DEF.

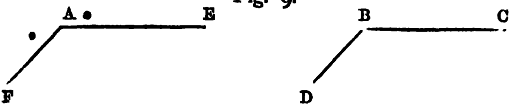
Corollary.—If FD and ED be produced to G and H, the lines DG and DH will be respectively parallel and in *opposite* directions to the lines AB and AC; therefore it follows that:

1. If two intersecting straight lines be respectively parallel and in opposite directions to two other intersecting straight lines, the angle contained by the first pair will be equal to the angle contained by the second pair, and the planes of these angles will be either coincident or parallel.

PROPOSITION 10.

Through a given point it is always possible to describe a plane parallel to a given plane.

Fig. 9.



Let A be the given point and DBC the given plane, then it shall be always possible to describe a plane through the point A parallel to the plane DBC.

Through the point B in the plane DBC draw two straight lines BC and BD.

Let AE and AF be two straight lines through the point A parallel to BC and BD respectively.

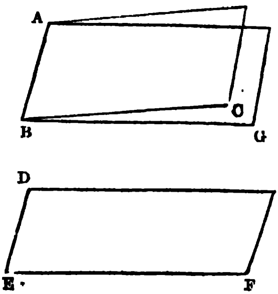
Because AE and AF are respectively parallel to the two intersecting straight lines BC and BD;

therefore the plane which contains AE and AF is parallel to the plane DBC.

PROPOSITION 11.

Only one plane can be described passing through a given point and parallel to a given plane.

Fig. 10.



Let A be the given point, and DEF the given plane, then only one plane can be described passing through the point A and parallel to the plane DEF.

If possible, let the two planes ABG and ABC, each parallel to DEF, pass through the point A, and let AB be their line of intersection.

In the plane DEF draw the

straight line EF not parallel to AB, and let any plane containing EF intersect the planes AC and AG in the straight lines BC and BG respectively.

Because the planes AC and DF are parallel, therefore the straight lines BC and EF, which are situated in these planes respectively, cannot meet, however far they may be produced.

But BC and EF are in the same plane,

therefore BC is parallel to EF.

Similarly, BG is parallel to EF,

therefore BC and BG are both parallel to EF,

which is impossible.

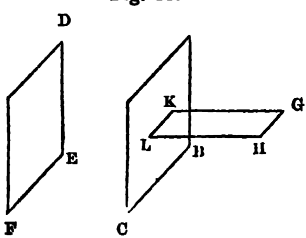
Corollary.—From the above reasoning it is evident that if two parallel planes be cut by the same plane their lines of intersection will be parallel.

PROPOSITION 12.

If two planes be parallel to one another, every straight line parallel to one of the planes shall be either situated in the other plane or be parallel to that plane.

Let ABC and DEF be two parallel planes, and let GH be a straight line parallel to the plane ABC, then GH shall be either situated in the plane DEF or be parallel to that plane.

Fig. 11.



Let the plane GL be described, containing the straight line GH, and let the line of intersection of this plane with the plane ABC be KL.

Then KL is parallel to GH (Bk. VII. Prop. 5).

Because the straight line KL is situated in the plane

ABC, and that the plane DEF is parallel to the plane ABC,

therefore KL is parallel to the plane DEF.

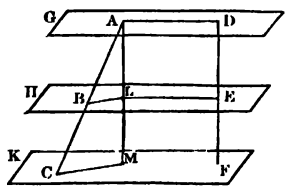
Because KL is parallel to the plane DEF, and GH is parallel to KL,

therefore GH is either situated in the plane DEF or is parallel to that plane (Bk. VII. Prop. 8).

PROPOSITION 13.

If two straight lines be cut by any number of parallel planes they shall be divided proportionally in the points in which they respectively meet the planes.

Fig. 12.



Let the two straight lines ABC and DEF be cut by the three parallel planes G, H, and K, in the points A, B, and C, D, E, and F, respectively, then shall the straight lines ABC and DEF be divided proportionally in these points respectively.

Through A draw the straight line ALM parallel to DEF, and let it meet the planes H and K in the points L and M respectively.

Join BL, CM, AD, and LE.

Because AL is parallel to DE, therefore they lie in the same plane.

Because the parallel planes G and H are cut by the plane containing AL and DE in the straight lines AD and LE respectively, therefore AD is parallel to LE (Bk. VII. Prop. 11, *Cor.*),

therefore AE is a parallelogram and AL is equal to DE.

Similarly LM is equal to EF.

Because AC and AM intersect in A, therefore they lie in

the same plane, and therefore BL is parallel to CM (Bk. VII. Prop. 11, *Cor.*).

Because BL is parallel to CM, therefore the straight lines AC and AM are divided proportionally in the points A, B, and C, A, L, and M, respectively (Bk. VI. Prop. 5).

But AL is equal to DE, and LM to EF,

therefore ABC and DEF are divided proportionally in the points A, B, and C, D, E, and F, respectively.

The same reasoning may be employed whatever the number of planes.

EXAMPLES.

1. Prove that one plane, and only one, can be drawn through a given straight line parallel to another given straight line.

2. Prove that through any given point one plane, and only one, can be drawn parallel to each of two given straight lines.

3. In every gauche quadrilateral, that is to say, one whose sides are not all situated in the same plane, the middle points of the sides are the angular points of a parallelogram.

4. Prove that two straight lines, which are each of them parallel to a given plane, and both of which meet two other straight lines situated in any manner in space, are not generally in the same plane.

5. Draw a straight line parallel to a given straight line, and meeting each of two other straight lines not situated in the same plane.

6. Draw a straight line of given length parallel to a given plane, and having its extremities on two given straight lines not in the same plane.

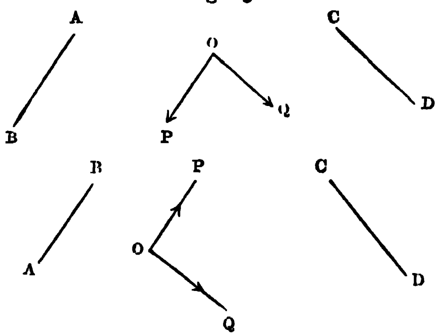
SECTION II.—ON THE PERPENDICULARS AND OBLIQUES TO PLANES.

DEFINITIONS.

81.—When two given straight lines do not intersect *their angle of inclination* is the angle contained by two straight lines drawn through any point respectively, parallel to, and in the same direction with, the given straight lines.

If this angle be a right angle, the two straight lines are said to be *perpendicular* or *at right angles to* one another.

Fig. 13.



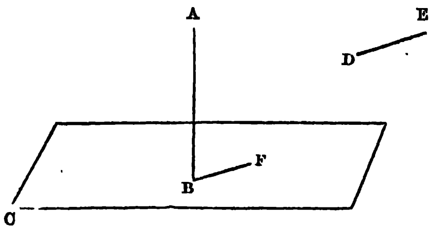
Thus, if AB and CD be two non-intersecting straight lines and OP and OQ be drawn through any point O parallel to AB and CD respectively, then POQ is said to be the angle between AB and CD.

82.—A straight line is said to be *perpendicular* to a given plane when it is perpendicular to every straight line that can be drawn in the given plane, and a straight line which is neither perpendicular nor parallel to a given plane is said to be *oblique* to that plane.

PROPOSITION 14.

If a straight line be perpendicular to a given plane it shall be also perpendicular to every straight line parallel to the given plane, and if a straight line be perpendicular to a given plane, every straight line perpendicular to the given straight line shall be parallel to the given plane.

Fig. 14.



Let the line AB be perpendicular to the plane C, and

1st. Let the straight line DE be parallel to this plane, then shall AB be perpendicular to DE.

Let AB meet the plane C in B, and through B draw BF parallel to DE, then BF lies in the plane C (Bk. VII. Prop. 8).

Because AB is perpendicular to, and BF is situated in, the plane C, therefore AB is perpendicular to BF (Def. 82).

Because AB is perpendicular to and DE is parallel to BF therefore AB is perpendicular to DE (Def. 81).

2nd. Let AB be perpendicular to DE, then shall DE be parallel to the plane C.

Through B draw BF parallel to DE.

Because the straight line AB is perpendicular to the plane C, therefore it is perpendicular to every straight line in that plane (Def. 82).

Because AB is perpendicular to, and BF is parallel to DE, therefore AB is perpendicular to BF (Def. 81);

therefore BF must lie in the plane C, otherwise two non-coincident straight lines might be drawn in the plane containing AB and BF, each of them perpendicular to AB.

Because BF lies in the plane C, and DE is parallel to BF, therefore DE is parallel to the plane C (Bk. VII. Prop. 6).

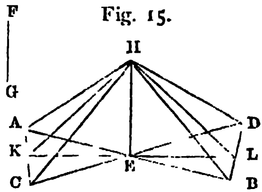
Corollary 1.—If two planes be parallel to one another, every straight line perpendicular to one of them will be also perpendicular to the other.

Corollary 2.—If a given straight line be perpendicular to a given plane, every straight line parallel to the given line will be also perpendicular to the given plane.

PROPOSITION 15.

If a straight line be perpendicular to each of two intersecting straight lines it shall be perpendicular to the plane which contains these two lines.

Fig. 15.



Let the straight line FG be perpendicular to each of the two straight lines AB and CD, which intersect in E, then shall FG be perpendicular to the plane which contains AB and CD.

Through E let EH be drawn parallel to FG, then by Def. 81, EH is perpendicular to each of the lines AB and CD.

Let AE, EB, CE, and ED be all equal to one another, and join HA, HB, HC, HD, AC, and BD.

Through E draw any straight line KEL in the plane containing AB and CD, let it meet AC and BD in the points K and L, and join HK and HL.

Because the triangles AEH, CEH, BEH, and DEH have the two sides and included angle in any one respectively equal to the two sides and included angle in any other,

therefore they are all equal to one another in all their parts and therefore AH, CH, BH, and DH are all equal.

Because the triangles AEC and BED have two sides and the included angle of one respectively equal to the two sides and the included angle of the other, therefore they are equal to one another in all their parts, and therefore AC is equal to BD, and ACE to BDE.

Because the triangles KEC and LED have the angles at E and C of the one respectively equal to the angles at E and D of the other, and the side CE equal to the side ED, therefore they are equal to one another in all their parts, and therefore KE is equal to LE.

Because the triangles AHC and BHD have the three sides AH, HC, and CA, of the one respectively equal to the three sides BH, HD, and DB of the other, therefore the angle ACH is equal to the angle BDH.

Because the triangles KCH and LDH have the sides KC, CH, and the included angle KCH of the one respectively equal to the sides LD, DH, and the included angle LDH of the other, therefore they are equal to one another in all their parts, and therefore KH is equal to LH.

And, finally, because the triangles KEH and LEH have the three sides KE, EH, and KH of the one respectively equal to the sides LE, EH, and LH of the other, therefore the angle KEH is equal to the angle LEH.

Therefore EH is at right angles to KL.

But FG is parallel to EH,

therefore FG is at right angles to KL.

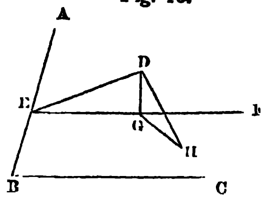
Similarly, FG is at right angles to every straight line in the plane containing AB and CD,

therefore FG is at right angles to the plane containing AB and CD (Def. 82).

PROPOSITION 16.

From a given point outside a given plane it is always possible to draw one straight line, and only one straight line, perpendicular to the given plane, and the straight line so drawn is shorter than any other straight line that can be drawn from the given point to the given plane.

Fig. 16.



Let AC be the given plane and D the given point without it, then

1st. It shall be always possible to draw one straight line, and only one straight line, from the point D perpendicular to the plane AC.

Take any straight line AB in the plane AC, and in the plane containing D and AB draw the straight line DE perpendicular to AB (Bk. I. Prop. 15).

Through the point E, and in the plane AC, draw EF perpendicular to AB (Bk. I. Prop. 11).

From the point D in the plane containing D and EF draw DG perpendicular to EF, then DG shall be perpendicular to the plane AC.

Because AB is perpendicular to the two intersecting straight lines DE and EG, therefore AB is perpendicular to the plane DEG (Bk. VII. Prop. 15),

therefore DG is perpendicular to AB.

Also DG was drawn perpendicular to EF,

therefore DG is perpendicular to each of the intersecting straight lines AB and EF situated in the plane AC,

therefore DG is perpendicular to the plane AC (Bk. VII. Prop. 15),

Also no other line can be drawn from D, besides DG, perpendicular to the plane AC.

For, if possible, let DH which meets the plane AC in H be perpendicular to that plane and join GH.

Then in the plane DGH the two straight lines DG and DH are each perpendicular to GH, which is impossible by Prop. 15, Bk. I.

2nd. DG is the shortest straight line that can be drawn from the point D to the plane AC.

For let DH, any other straight line from the point D to the plane AC, meet this plane in H and join HG.

Because DG is perpendicular to the plane AC, therefore DG is perpendicular to the line HG, and DH is any other straight line from D to HG,

therefore DG is less than DH (Bk. I. Prop. 15).

Corollary 1.—If from any point without a plane two straight lines be drawn, one perpendicular to the plane and the other perpendicular to any straight line in that plane, then the straight line joining the feet of these perpendiculars will be perpendicular to the straight line in the plane.

Corollary 2.—If two straight lines be each of them perpendicular to a given plane they shall be parallel to one another, for if they meet they will lie in one plane, and then there will be two intersecting straight lines in this plane each perpendicular to the line joining the points of intersection of the given straight lines with the given plane.

Corollary 3.—If a straight line be drawn from a point in a given plane parallel to a perpendicular to that plane from a point without it, then the former line will be also perpendicular to the given plane, and therefore one line, and only one line, may always be drawn from a given point in a given plane perpendicular to that plane.

PROPOSITION 17.

It is always possible to describe one plane, and one plane only, passing through a given point and perpendicular to a given straight line.

Let A be the given point and BC the given straight line, then it shall be always possible to describe one plane, and

one plane only, passing through the point A and perpendicular to the straight line BC.

First, let the point A be in the straight line BC (Fig. 17).

Fig. 17.

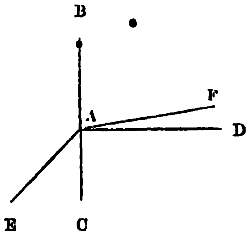
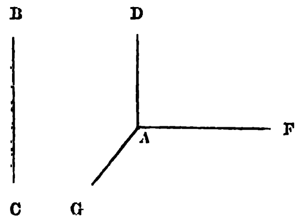


Fig. 18.



Through A in any two planes each containing BC draw the straight lines AD and AE perpendicular to BC (Bk. I. Prop. 11),

then the plane which contains AD and AE is perpendicular to AB (Bk. VII. Prop. 15).

Also no other plane passing through A can be perpendicular to AB.

If possible, let another plane containing the point A be perpendicular to BC, and since this plane cannot contain both AD and AE, let it intersect the plane BAD in the line AF different from one of these lines as AD.

Because BC is perpendicular to, and AF lies in, this plane, therefore BAF is a right angle.

Also BAD is a right angle,

therefore BAF is equal to BAD,

and this is impossible since AF, AD, and AB are in one plane.

Next let the point A be situated out of the line BC (Fig. 18).

Through A draw AD parallel to BC.

Then by the first part a plane AGF may be described passing through the point A and perpendicular to the line AD.

Because AD is perpendicular to the plane FAG, and BC is parallel to AD,

therefore BC is perpendicular to the plane FAG
(Bk. VII. Prop. 14, *Cor.* 2),

and it may be proved, as in the last case, that no other plane besides FAG can be drawn through the point A perpendicular to the straight line AB.

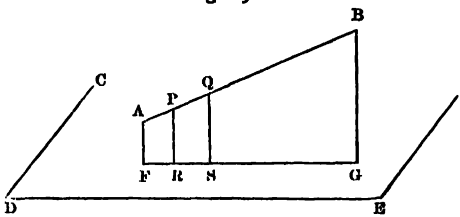
DEFINITION.

83.—The foot of the perpendicular drawn from a given point outside a given plane to the plane is called the *projection of that point upon the plane*, and the line which passes through the projections of all the points of a given line, whether straight or curved, is called the *projection of that line upon the plane.*

PROPOSITION 18.

The projection of a given straight line upon a given plane is another straight line.

Fig. 19.



Let AB be the given straight line and CDE the given plane.

Let the projections of the successive points of the line AB, as A, P, Q, &c., B, upon the plane CDE, be F, R, S, &c., G.

Because AF and PR are parallel straight lines, and A and P are points upon the straight line AB, therefore AF and PR lie in one plane passing through AB and PR.

Similarly, PR and QS lie in one plane passing through AB and PR,

therefore AF, PR, and QS lie in one plane passing through AB (Bk. VII. Prop. 2);

therefore all the perpendiculars upon CDE from different points in AB lie in one plane passing through AB;

therefore all the points F, R, S, &c., lie in the intersection of a plane passing through AB with the plane CDE;

therefore all these points are situated in a straight line (Bk. VII. Prop. 4).

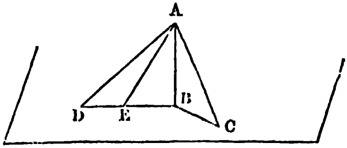
Corollary.—If AB be parallel to CDE the projection will be a straight line equal and parallel to AB.

PROPOSITION 19.

If two obliques be drawn to a given plane from a given point without it making unequal angles with the perpendicular to the plane drawn from that point, that oblique which makes the smaller angle with this perpendicular shall be less than the other. And if from a given point without a given plane two unequal obliques be drawn to that plane, the angle between the shorter of these two obliques and the perpendicular to the plane from the external point shall be less than the angle between the longer oblique and that same perpendicular.

Let A be the given point and CBD the given plane, and from A let the straight lines AB, AC, and AD be drawn to the plane, AB being perpendicular and AC and AD oblique to this plane, then

Fig. 20.



1st. If the angle CAB be less than the angle DAB, the line AC shall be less than the line AD.

Join DB and CB.

Because DAB is greater than CAB let AE be drawn in

the plane DAB, so that the angle EAB is equal to the angle CAB.

Because EAB is equal to CAB, and EBA to CBA, therefore the two triangles EAB and CAB have two angles of the one respectively equal to two angles of the other, and the side AB common to both,

therefore they are equal to one another in all their parts,

and therefore AE is equal to AC.

Because AB is perpendicular to BD, and EAB less than DAB, therefore AE is less than AD (Bk. I. Prop. 15);

therefore also AC is less than AD.

2nd. If AC be less than AD the angle CAB shall be less than the angle DAB.

Because AC is less than AD and greater than AB, therefore the circle described in the plane ABD with centre A and radius AC will cut DB in some point E between D and B (Bk. II. Prop. 3).

Join AE.

Because the two right-angled triangles EAB and CAB have the hypothenuse EA and side AB of the one equal respectively to the hypothenuse CA and side AB of the other, therefore the two triangles are equal in all their parts,

therefore EAB is equal to CAB,

therefore CAB is less than DAB.

PROPOSITION 20.

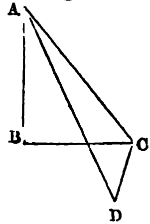
The angle which any oblique to a given plane makes with its projection on that plane is less than the angle which the same oblique makes with any other straight line in that plane.

Let BCD be the given plane and AC any oblique to it: from A draw AB perpendicular to the plane BCD and join BC so that BC is the projection of AC upon BCD, then if any straight line CD be drawn through the point C in the

plane BCD the angle ACB shall be less than the angle ACD.

Make CD equal to CB and join AD.

Fig. 21.



Because AB is perpendicular and AD is oblique to the plane BCD,

therefore AB is less than AD.

Because in the two triangles ACB and ACD the two sides AC and CB are respectively equal to the two sides AC and CD, and the side AB is less than the side AD,

therefore the angle ACB is less than the angle ACD.

DEFINITION.

84.—*The angle of inclination of a straight line to a plane* is the angle between the line and its projection upon the plane.

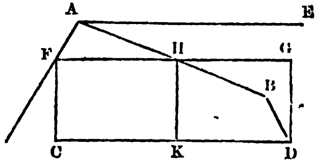
PROPOSITION 21.

If there be two given straight lines not situated in the same plane, then it shall be always possible to draw one straight line, and one only, meeting each of the given straight lines at right angles, and this line shall be the shortest distance between the two given straight lines.

Let AB and CD be the two given straight lines not lying in the same plane, then

1st. It shall be possible to draw one straight line, and one only, meeting each of the lines AB and CD at right angles.

Fig. 22.



Through A, any point in AB, draw AE parallel to CD, and let FG be the projection of CD upon the plane containing AE and AB.

Because CD is parallel to AE, therefore CD is parallel to the plane EAB (Bk. VII. Prop. 6),

therefore FG is parallel to CD (Bk. VII. Prop. 18, *Cor.*),
therefore FG is also parallel to AE (Bk. VII. Prop. 7, *Cor.*),
and therefore FG intersects AB in some point H.

From H draw HK perpendicular to the plane EAB,
then HK shall meet both AB and CD at right angles.

For by construction HK passes through a point in AB and is perpendicular to AB, i.e. HK meets AB at right angles.

Because FG is the projection of the straight line CD upon the plane EAF, therefore the straight line HK drawn through a point H in FG perpendicular to the plane EAF must meet the straight line CD in some point as K (Def. 83).

Because CD is parallel and HK perpendicular to the plane EAB, therefore HK is perpendicular to CD (Prop. 14),
therefore HK meets both of the straight lines AB and CD at right angles.

Also no other straight line besides HK can meet both AB and CD at right angles.

For every straight line perpendicular to AB and CD must be perpendicular to AB and AE, and therefore to the plane EAB.

And the perpendiculars from every point in CD to the plane EAB must meet that plane in the line FG (Def. 83).

Therefore every straight line which meets both AB and CD at right angles must pass through the intersection of AB and FG, that is, through the point H.

But only one line can be drawn through the point H perpendicular to the plane EAB, and since HK is such a line, it follows that HK is the only straight line which can meet both AB and CD at right angles.

2. HK shall be the shortest straight line that can be drawn between the two given lines AB and CD.

Let BD be any other straight line between the two given lines AB and CD, and let DG be perpendicular to the

plane EAB, then DGHK is a parallelogram, and DG is therefore equal to HK.

Because DG is perpendicular and DB oblique to the plane EAF, therefore DB is greater than DG,

therefore DB is greater than HK.

EXAMPLES.

1. If a straight line meet a plane in a certain point so as to make equal angles with three straight lines drawn in the plane and through that point, prove that the straight line will be perpendicular to the plane.

2. From a given point outside a given plane a perpendicular is drawn to the plane, and also a number of obliques all equally inclined to the plane. Show that the length of these obliques are all equal, and that their points of intersection with the plane lie in a circle whose centre is the foot of the perpendicular.

3. By the law of optics, if a ray of light be incident upon a plane reflecting surface the reflected ray and the incident ray lie in the same plane with the perpendicular to the plane at the point of incidence and make equal angles with this perpendicular on opposite sides of it. Prove that if any two points be taken, one on the incident and one on the reflected ray, the sum of the distance of these points from the point of incidence will be less than the sum of their distances from any other point in the plane.

4. Prove also that if a pencil of parallel rays be incident on a plane reflecting surface the reflected pencil will also consist of parallel rays.

5. When a ray is refracted at a plane surface, the refracted ray and the incident ray lie in the same plane with the perpendiculars to the surface at the point of incidence, and make such angles with it that if any two points be taken on the perpendicular to the plane through the point of incidence on opposite sides of that point and equidistant from it the

distances of these points from the two rays are in a constant ratio. Prove that if a pencil of parallel rays be incident on a plane surface the refracted pencil also consists of parallel rays.

6. Prove that the locus of points on a plane at which a given straight line subtends a right angle is a circle whose centre is the foot of the perpendicular let fall from the middle point of the line upon the plane.

7. If a plane be drawn parallel to two non-intersecting straight lines through the middle point of the line which meets each of them at right angles, prove that it passes through the middle points of all the lines joining any point in one line with any point in the other.

8. Find the point in a given straight line which is equidistant from any two given points.

9. Find the locus of points such that their distances from two parallel planes may be always to each other in a given ratio.

10. AB and AC are two given straight lines intersecting in A. Find the locus of the point P, such that the sum of the projections of AP upon AB and AC may be equal to a given line.

11. If a straight line be parallel to a plane, prove that its shortest distances from all lines in the plane which are not parallel to itself are of equal length.

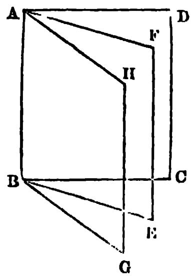
12. Find the locus of the middle point of a straight line of constant length, whose extremities lie one on each of two straight lines perpendicular to one another and not situated in the same plane.

Section III.—On Dihedral Angles.

DEFINITION.

85.—When two planes intersect, the amount or quantity of turning about their line of intersection which is required to bring one of them into coincidence with the other is called the *dihedral angle* between the planes.

Fig. 23.



Thus let AB be the line of intersection of the two planes AC and AE, then the amount of turning about AB which is required to bring one of these planes, as AC, into coincidence with the other, AE, is called the dihedral angle between the planes AC and AE. This angle is expressed by the four letters DABE, the two middle letters A and B being points in the line of intersection, and the extreme letters D and E being points in each plane respectively.

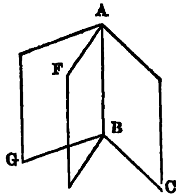
Note.—It is to be observed that, as in the case of the angle between two straight lines, there are an infinite number of dihedral angles between two given planes. However, when the dihedral angle is spoken of without further description it is always to be understood that the smallest possible angle is contemplated.

DEFINITION.

86.—When two dihedral angles have a common edge, and the intermediate face common to both, they are said to be

adjacent; thus the angles FABG and FABC are adjacent angles.

Fig. 24.



DEFINITION.

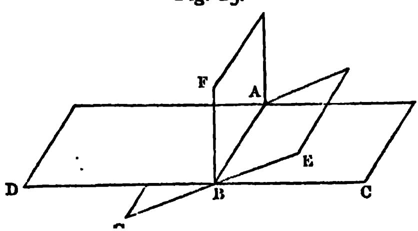
87.—Two dihedral angles are said to be *equal* to one another, when one of them can be applied to the other so that the edges coincide, and the two plane faces of the one coincide respectively with the two plane faces of the other.

Note.—It must be remembered that this test of equality applies only to the smallest possible dihedral angles.

DEFINITION.

88.—One plane is said to be *perpendicular* or *at right angles to* another plane when the adjacent dihedral angles which the second plane forms with the first on opposite sides of it are equal to one another.

Fig. 25.

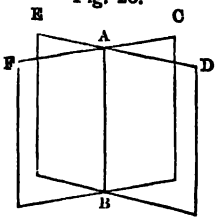


Thus if DAC and FBA be two planes intersecting in AB, and such that the adjacent dihedral angles DBAF and CBAF are equal to one another, the plane FBA is said to be perpendicular to the plane DAC, and each of the equal adjacent dihedral angles is called a *right dihedral angle.*

DEFINITION.

89.—If two planes cut each other, the dihedral angles between the opposite parts of the planes are called opposite dihedral angles with the same edge.

Fig. 26.



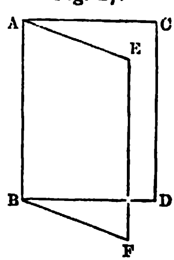
Thus the dihedral angles EABF and CABD are opposite dihedral angles with the same edge.

PROPOSITION 22.

If two points be taken in the line of intersection of two planes, and through each of these points two straight lines be drawn perpendicular to the line of intersection, one line in each plane respectively, then the angle between one pair of these lines shall be equal to that between the other pair, and the planes of these angles shall be each of them perpendicular to the line of intersection of the two planes.

Let AD and AF be two planes intersecting in the line AB.

Fig. 27.



Through the point A draw AC and AE each perpendicular to AB, and situated in the planes AD and AF respectively, and let BD and BF be similarly drawn through the point B.

Then shall the angles CAE and DBF be equal to each other, and the planes of these angles shall be each perpendicular to the straight line AB.

Because the straight lines CA, AB, and BD are situated in one plane, and the interior angles CAB and ABD are together equal to two right angles, therefore AC is parallel to BD.

Similarly, AE is parallel to BF,

therefore the angle CAE is equal to the angle DBF (Bk. VII. Prop. 9).

Again, because AB is perpendicular to each of the intersecting straight lines AC and AE, therefore it is perpendicular to the plane CAE.

And for a similar reason AB is likewise perpendicular to the plane DBF.

DEFINITION.

90.—If a point be taken in the line of intersection of two planes, and two straight lines be drawn through this point, one in each plane, and each perpendicular to their line of intersection, the angle between these straight lines is called the *plane angle corresponding to the dihedral angle between the planes.*

PROPOSITION 23.

If two dihedral angles be equal to one another the corresponding plane angles shall be also equal to one another, and if two plane angles be equal to one another the corresponding dihedral angles shall be also equal to one another.

Fig. 28.

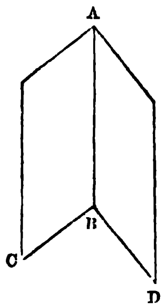
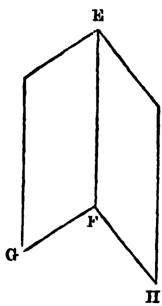


Fig. 29.



1st. Let CBAD and GFEH be two equal dihedral angles and let CBD and GFH be the corresponding plane angles at the points B and F, respectively, then shall CBD be equal to GFH.

Apply CBAD to GFEH, so that the point B coincides with the point F, the straight line BA with the straight line FE, and the plane AC with the plane EG, and let both angles be situated on the same side of the plane EG.

Because the angles CBAD and GFEH are equal to one another, and the plane faces AC and EG are coincident, therefore also the plane faces AD and EH will be coincident (Def. 87).

Because the point B coincides with the point F, the plane AD with the plane EH, and the straight line AB with the straight line EF, and because the angle ABD is equal to the angle EFH, therefore the straight line BD coincides with the straight line FH.

Because BC coincides with FG, and BD with FH, therefore the angle CBD is equal to the angle GFH.

2nd. If the plane angle CBD be equal to the plane angle GFH, then the dihedral angle CBAD shall be equal to the dihedral angle GFEH.

Apply the dihedral angle CBAD to the dihedral angle GFEH so that the point B may coincide with the point F, the straight line BC with the straight line FG, and the plane CBD with the plane GFH, and let both angles lie on the same side of the plane EG.

Because B coincides with F, and BC with FG, and the angle CBD is equal to the angle GFH, therefore BD coincides with FH.

Because the point B coincides with the point F, and the plane CBD with the plane GFH, therefore AB, the perpendicular to the plane CBD, coincides with EF, the perpendicular to the plane GFH.

Because AB coincides with EF, and BC with FG, therefore the plane AC coincides with the plane EG.

Similarly the plane AD coincides with the plane EH, therefore by Definition 87 the dihedral angle CBAD is equal to the dihedral angle GFEH.

PROPOSITION 24.

The ratio of any two dihedral angles is equal to the ratio of the corresponding plane angles.

Since equal plane angles correspond to equal dihedral angles it may be proved, exactly as in Props. 1 and 2 of Bk. VI., that if two commensurable plane angles and the corresponding dihedral angles be given, and if one dihedral angle and its corresponding plane angle be divided into the same number of aliquot parts, then the second dihedral angle and second plane angle will be equimultiples of each of the aliquot parts of the first dihedral angle and first plane angle respectively, whence the proposition is proved to be true for commensurable, plane, and dihedral angles.

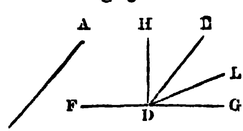
The Proposition may then be extended to incommensurable angles by reasoning similar to that employed in the propositions above mentioned.

PROPOSITION 25.

Through any straight line in a given plane it is always possible to describe one plane, and only one plane, perpendicular to the given plane, and the sum of the adjacent dihedral angles made by any plane with another plane on which it stands are together equal to two right dihedral angles.

1st. Let ABC be the given plane, and DE any straight line in it, then it shall be always possible to describe a plane containing the straight line DE and perpendicular to the plane ABC.

Fig. 30.



Through the point D and in the plane ABC draw the straight line FDG at right angles to DE.

Through the point D, and in the plane passing through FG perpendicular to DE, draw the straight line DH perpendicular to FG, then HDF and HDG are the plane angles corresponding to the dihedral angles FDHE and GDHE.

Because HD is perpendicular to FG, therefore the angle FDH is equal to the angle GDH ;

therefore also the dihedral angle FDHE is equal to the dihedral angle GDHE (Bk. VII. Prop. 23) ;

therefore the plane containing HD and DE is perpendicular to the plane ABC.

2nd. Let LD be the line of intersection of any other plane containing DE with the plane HDG.

Because ED is perpendicular to the plane HDG, therefore ED is also perpendicular to DL ;

therefore LDF and LDG are the plane angles corresponding to the adjacent dihedral angles FDEL and GDEL ;

therefore $\frac{\text{FDL}}{\text{FDH}}$ is equal to $\frac{\text{FDEL}}{\text{FDEH}}$ (Bk. VII. Prop. 24) ;

and $\frac{\text{GDL}}{\text{FDH}}$ is equal to $\frac{\text{GDEL}}{\text{FDEH}}$,

therefore $\frac{\text{FDL}+\text{GDL}}{\text{FDH}}$ is equal to $\frac{\text{FDEL}+\text{GDEL}}{\text{FDEH}}$.

But FDL+GDL is equal to 2FDH,

therefore FDEL+GDEL is equal to 2FDEH ;

that is, FDEL and GDEL are together equal to two right dihedral angles.

Corollary 1.—The corresponding plane angle to a right dihedral angle is a right angle, and the corresponding dihedral angle to a right angle is a right dihedral angle.

Corollary 2.—Opposite dihedral angles with the same side are equal to one another.

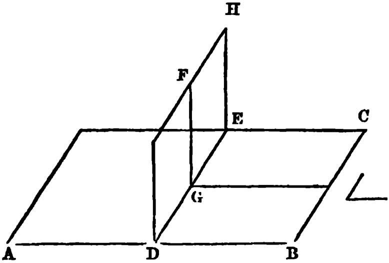
PROPOSITION 26.

If two planes be perpendicular to one another every straight line drawn in one of the planes perpendicular to their line of intersection shall be perpendicular to the other plane.

Let the two planes ABC and DEH, intersecting in DE, be perpendicular to one another, and let FG be drawn in

the plane DH perpendicular to DE, then FG shall be perpendicular to the plane AC.

Fig. 31.



Through the point G, in the plane AC, draw GL perpendicular to DE.

Because GF and GL are each perpendicular to DE, therefore the angle FGL is the plane angle corresponding to the dihedral angle FGEL (Def. 90).

Because the dihedral angle FGEL is a right dihedral angle,
therefore also the angle FGL is a right angle (Bk. VII. Prop. 25, *Cor.* 1);
therefore FG is perpendicular to each of the intersecting straight lines DE and GL, lying in the plane AC;
therefore FG is perpendicular to the plane AC (Bk. VII. Prop. 15).

PROPOSITION 27.

If a straight line be perpendicular to a given plane, then every plane which contains this straight line, or is parallel to it, shall be also perpendicular to the given plane.

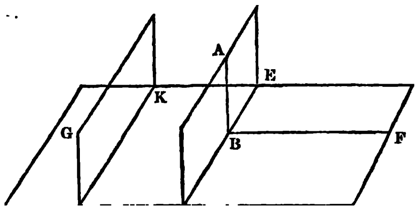
Let the straight line AB be perpendicular to the plane DCE, and

1st. Let ABC be a plane containing AB, then ABC shall be perpendicular to DCE.

Let the planes ABC and DCE intersect in the straight line EBC.

Through the point B, and in the plane DCE, draw the straight line BF perpendicular to BC, then BF is also perpendicular to AB (Def. 82).

Fig. 32.



Because ABC and FBC are each of them right angles, therefore ABF is the plane angle corresponding to the dihedral angle ABCF.

Because the plane angle ABF is a right angle, therefore also the dihedral angle ABCF is a right dihedral angle, or the plane ABC is perpendicular to the plane DCE.

2nd. Let GK be a plane parallel to AB.

Draw any straight line GH in the plane GK parallel to AB.

Because AB is perpendicular to the plane DCE, and GH is parallel to AB, therefore GH is perpendicular to the plane DCE (Bk. VII. Prop. 14, *Cor.* 2);

therefore the plane GK, containing GH, is perpendicular to the plane DCE by the first case.

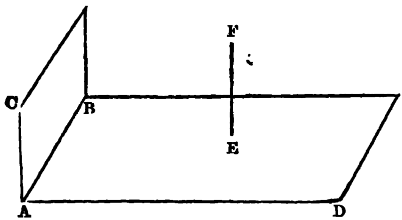
PROPOSITION 28.

If one plane be perpendicular to another plane, then every straight line perpendicular to the former plane shall be either parallel to the latter plane or be situated in that plane.

Let the plane BAD be perpendicular to the plane BAC, and let EF be a straight line perpendicular to BAD, then

EF shall be either parallel to the plane BAC or be situated in that plane.

Fig. 33.



If EF be situated in the plane BAC the proposition is proved, but if not, either the straight line EF and the plane BAC must intersect when produced, or else EF must be parallel to the plane BAC.

But EF and the plane BAC cannot intersect, for if they did, then the straight line drawn from their point of intersection and in the plane BAC perpendicular to AB would be perpendicular to the plane BD, and therefore there would be two intersecting straight lines, each perpendicular to the same plane, which is impossible (Bk. VII. Prop. 16, *Cor.* 2),

therefore the straight line EF is parallel to the plane BAC.

Corollary.—If two planes be, each of them, perpendicular to a third plane, their line of intersection will be also perpendicular to this third plane.

PROPOSITION 29.

Through any straight line oblique to a given plane, one plane, and one plane only, can be described perpendicular to the given plane.

Let AB be the given straight line oblique to the given plane CDE, then one plane, and only one plane, can be described containing the straight line AB and perpendicular to the plane CDE.

Take any point A in AB, and draw AF perpendicular to the plane CDE.

Because AF is perpendicular to the plane CDE, therefore every plane passing through AF is also perpendicular to CDE.

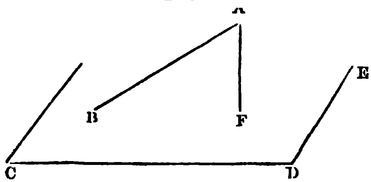


Fig. 34.

But a plane can be described containing AB and AF (Bk. VII. Prop. 1, *Cor.*),

therefore a plane can be described containing AB, and perpendicular to the plane CDE.

Also no more than one such plane can be described.

Because AF is perpendicular to the plane CDE, therefore every plane perpendicular to CDE either contains AF or is parallel to AF.

If it contains AF and also AB, it is coincident with the plane already found (Bk. VII. Prop. 1, *Cor.*).

If it be parallel to AF it cannot contain AB which intersects AF,

therefore only one plane can be described containing the straight line AB and perpendicular to the plane CDE.

Examples.

1. Through a point O two straight lines OA and OB are drawn, each parallel to a given plane, and through the same point O two planes are drawn perpendicular to OA and OB respectively. Prove that the line of intersection of these two planes will be perpendicular to the given plane.

2. If a point be projected upon each of two intersecting planes, prove that the straight lines drawn from the two projections on the planes perpendicular to the line of intersection of these planes will meet this line of intersection in the same point.

3. If the projections of any line upon two intersecting planes be each of them straight lines, prove that the line so projected must be also a straight line.

4. Find the locus of the points in space equidistant from two given intersecting straight lines.

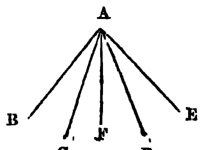
5. If a number of planes be drawn so that the lines of intersection of all of them, taken two and two, are parallel to the same straight line, prove that the perpendiculars dropped upon these planes from any point whatever all lie in one and the same plane.

SECTION IV.—ON POLYHEDRAL ANGLES.

DEFINITION.

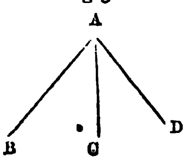
91.—When several straight lines meet in a point they are said to form a *polyhedral angle*, the point in which the lines meet is called the *summit*, the several straight lines are called the *edges*, and the plane angles of inclination of successive straight lines are called the *faces*.

Fig. 35.



Thus the straight lines AB, AC, AD, AE and AF (Fig. 37) are said to form the polyhedral angle ABCDEF; the lines AB, AC, &c., are called the edges; and the angles BAC, CAD, &c., are called the faces of the polyhedral angle.

Fig 36.



The polyhedral angle at A is said to be contained by the edges AB, AC, AD, AE, and AF.

When there are only three edges, as AB, AC, and AD (Fig. 38), the angle at A is called a trihedral angle.

PROPOSITION 30.

In every polyhedral triangle any one face is less than the sum of all the rest.

First let ABCD be any trihedral angle at A contained by the edges AB, AC, and AD, and let the face BAD be greater

Fig. 37.

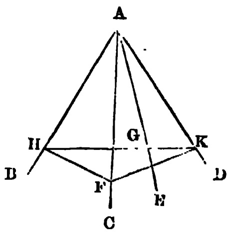
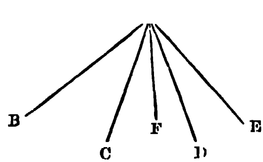


Fig. 38.



than either of the other two faces BAC and CAD.

Since BAD is greater than BAC let BAE in the plane BAD be made equal to BAC, in AE take any point G, and from AC cut off AF equal to AG.

Through G in the plane BAD draw the straight line HGK meeting AB and AD in H and K, and join FH and FK.

Because HA, AG, and the included angle HAG are respectively equal to HA, AF, and the included angle HAF, therefore the two triangles HAG and HAF are equal to one another in all their parts, and therefore HG is equal to HF.

Because the two sides HF and FK of the triangle HFK are together greater than HK, and that HF is equal to HG, therefore FK is greater than GK.

Because in the two triangles FAK and GAK the two sides FA and AK are respectively equal to the two sides GA and AK, but the base FK is greater than the base GK, therefore the angle FAK is greater than the angle GAK.

But the angle HAF is equal to the angle HAG,
therefore the two angles HAF and FAK are together greater than the two angles HAG and GAK,
that is, the angles BAC and CAD are together greater than the angle BAD.

Next, let the polyhedral angle ABCDEF be contained by the five edges AB, AC, AD, AE, and AF.

By what has been already proved, the angles BAC and CAD are together greater than the angle BAD.

Similarly, the angles BAD and DAE are together greater than the angle BAE,
therefore the angles BAC + CAD + DAE are together greater than the angle BAF,
and similarly, the angles BAC + CAD + DAE + EAF are together greater than the angle BAF.

And the same is true whatever be the number of edges of the given polyhedral angle.

DEFINITION.

92.—A polyhedral angle is said to be *convex* when the portion of space bounded by the planes of the faces is situated wholly on the *same* side of any one of these planes when produced indefinitely; in other cases it is said to be *concave*. See Figs. 39 and 40.

Fig. 39.

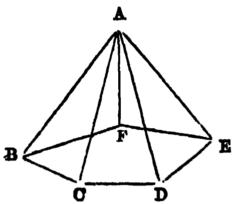
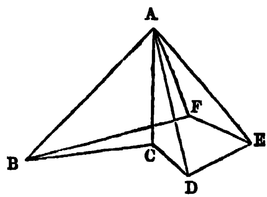


Fig. 40.

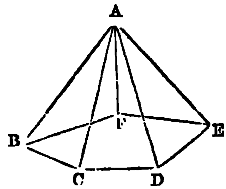


PROPOSITION 31.

In any convex polyhedral angle the sum of the faces is together less than four right angles.

Let A be any convex polyhedral angle contained by the five edges AB, AC, AD, AE, and AF, then shall the sum of the faces BAC, CAD, DAE, EAF, and FAB be together less than four right angles.

Fig. 41.



Let any plane BCDEF be described so as to cut each of the edges in the points B, C, D, E, and F, and join BC, CD, DE, EF, and FB.

Because the trihedral angle at B is contained by the edges BA, BF, and BC, therefore FBC is less than ABF+ABC.

Similarly, BCD is less than ACB+ACD,

and so on for every other angle of the polygon BCDEF; therefore the sum of all the angles of the polygon BCDEF is less than the sum of all the angles at the bases of the triangles ABC, ACD, and so on.

But the sum of all the angles of the polygon, together with four right angles, is equal to twice as many right angles as there are sides of the polygon.

Also, the sum of all the angles at the bases of the triangles, together with the sum of all the angles at the vertices of the triangles, is also equal to twice as many right angles as the figure has sides,

therefore the sum of all the angles at the vertices is less than four right angles;

that is, the sum of the faces of the polyhedral angle at A is less than four right angles.

And the same is true whatever be the number of edges of the given polyhedral angles.

EXAMPLES.

1. In every trihedral angle the planes bisecting the three trihedral angles all pass through the same straight line.

2. In every trihedral angle the planes which bisect the plane angles, and are perpendicular to the corresponding plane faces, all pass through the same straight line.

3. In every trihedral angle the planes drawn through the edges perpendicular to the opposite faces all pass through the same straight line.

MISCELLANEOUS EXAMPLES.

1. If a straight line be equally inclined to the two faces of any dihedral angle, prove that the points in which it meets the faces containing this angle will be equally distant from the time of insertion of these faces.

2. Find the locus of points equidistant from two intersecting planes.

3. Prove that if through the same point of the edge of any dihedral angle a straight line be drawn in each plane face making any given angle with that edge, then the plane angle between the two straight lines thus drawn will not vary proportionally to the dihedral angle, unless the said given angle be a right angle.

4. If a plane be drawn through either diagonal of a parallelogram, prove that the perpendiculars drawn to this plane from the opposite extremities of the other diagonal will be equal to one another.

5. Let the straight line AB be divided at C so that the ratio $\frac{AC}{BC}$ may be equal to $\frac{m}{n}$ where m and n represent any whole numbers, and from A, B, and C draw AD, BE, and CF perpendicular to any plane whatever. Prove that $(m+n)$ CF is equal to n . AD + m BE.

6. If the sum of the perpendiculars dropped from any

point A upon two planes be equal to the sum of the perpendiculars dropped from another point B on the same planes, and the straight line AB be joined, prove that the sum of the perpendiculars dropped from any point whatever in AB upon the two planes will be equal to the aforesaid sum from either A or B

7. If there be three points A, B, and C, such that the sum of the two perpendiculars dropped from these points upon any two planes is the same for each point, prove that this will also be true of any point whatever in the plane passing through A, B, and C.

8. Find the locus of a point, such that the sum of its distance from two given planes may be equal to a given straight line.

9. Find the locus of a point such that the sum of its distances from three given planes may be equal to a given straight line.

10. Lines are drawn through any trihedral angle, one in each of the three planes containing that angle, and each perpendicular to the edge opposite to that face. Prove that the three lines thus drawn lie in the same plane.

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